MULTIVARIABLE CALCULUS
EXAM 2 - PART 1
FALL 2019

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems - each is worth 10 points. Justify all answers/solutions. Calculators are not permitted. Best of luck.

(1) A particle is travelling along the path $\mathbf{x}(t) = (\cos(e^t), 3 - t^2, t)$ when suddenly at $t = 1$ all forces acting upon it cease. What happens to the particle? That is, find the tangent line to the path at the time $t = 1$.

We may find this line as follows:

First at $t=1$, the particle is at $\vec{x}(1) = (\cos(e), 2, 1)$.

The velocity along the path is given by

$$\vec{x}\,'(t) = (-e^t \sin(e^t), -2t, 1)$$

and at $t=1$ this is

$$\vec{x}\,'(1) = (-e \sin e, -2, 1).$$

So the tangent line is

$$\vec{\ell}(t) = (\cos(e), 2, 1) + (-e\sin(e), -2, 1)(t-1)$$

$$= (\cos(e) + e\sin(e) - te\sin(e), -2t+4, t).$$

Date: November 11, 2019.

(2) Calculate the length of the path $\mathbf{x}(t) = (t^2, \frac{2}{3}(2t+1)^{3/2})$ for $0 \leq t \leq 4$.

Arc-length is given by the following integral:

$$L(\vec{x}(t)) = \int_a^b \|\vec{x}\,'(t)\|\, dt.$$

For this example, $\vec{x}\,'(t) = (2t, 2(2t+1)^{1/2})$.

Thus,

$$L(\vec{x}) = \int_{t=0}^{t=4} \sqrt{(2t)^2 + 4(2t+1)}\, dt$$

$$= \int_0^4 \sqrt{4t^2 + 8t + 4}\, dt$$

$$= \int_0^4 2\sqrt{t^2+2t+1}\, dt = \int_0^4 2\sqrt{(t+1)^2}\, dt$$

$$= \int_0^4 2|t+1|\, dt = 2\int_0^4 t+1\, dt = 2\left(\frac{t^2}{2} + t\right)\Big|_0^4$$

$$= 24$$

(3) Calculate the flow line $\mathbf{x}(t)$ for the vector field $\mathbf{F}(x, y, z) = 2\mathbf{i} - 3y\mathbf{j} + z^3\mathbf{k}$ that passes through the point $\mathbf{x}(0) = (3, 5, 7)$.

A flow line of a vector field $\vec{F}$ is a path $\vec{x}(t)$ s.t $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$

So if $\vec{x}(t) = (x(t), y(t), z(t))$, then $\vec{x}\,'(t) = \vec{F}(x, y, z) = (2, -3y, z^3)$

We obtain:

(1) $x'(t) = 2$ and so $x(t) = 2t + C$

From the initial condition: $x(0) = 3 = 2 \cdot 0 + C \Rightarrow C = 3$

So $x(t) = 2t + 3$

(2) $y'(t) = -3y$. This is a separable differential equation, which we solve as follows

$$\frac{dy}{dt} = -3y \Rightarrow \frac{dy}{y} = -3\,dt \Rightarrow \int \frac{dy}{y} = \int -3\,dt$$

$$\Rightarrow \ln|y| = -3t + C \Rightarrow y = e^{-3t + C} = C_1 e^{-3t}$$

From the initial condition: $y(0) = 5 = C_1 e^0 \Rightarrow C_1 = 5$

$\Rightarrow y(t) = 5e^{-3t}$

(3) $z'(t) = z^3$. This is a separable differential equation, which we solve as follows.

$$\frac{dz}{dt} = z^3 \Rightarrow \frac{dz}{z^3} = dt \Rightarrow \int \frac{dz}{z^3} = \int dt \Rightarrow -\frac{1}{2} \cdot \frac{1}{z^2} = t + C$$

$$\Rightarrow \frac{1}{z^2} = -2t + C_1 \Rightarrow z^2 = \frac{1}{-2t + C_1} \Rightarrow z = \frac{1}{\sqrt{-2t + C_1}}$$

From the initial condition $z(0) = 7 = \frac{1}{\sqrt{C_1}} \Rightarrow C_1 = 1/49$

$\Rightarrow \vec{x}(t) = (2t + 3, 5e^{-3t}, \frac{1}{\ldots})$

(4) For the vector field $\mathbf{F} = (\cos(yz) - x)\mathbf{i} + (\cos(xz) - y)\mathbf{j} + (\cos(xy) - z)\mathbf{k}$, find:

(a) the divergence

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= -1 + -1 + -1 = -3.$$

(b) the curl.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(yz)-x & \cos(xz)-y & \cos(xy)-z \end{vmatrix}$$

$$= (-\sin(xy)x + \sin(xz)x)\vec{i} - (-\sin(xy)y + \sin(yz)y)\vec{j}$$

$$+ (-\sin(xz)z + \sin(yz)z)\vec{k}$$

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(1) For $f(x, y, z) = e^{2x-3y}\sin(5z)$ and the point $\mathbf{a} = (0, 0, 0)$, express the second order Taylor polynomial $p_2(x, y, z)$, using the derivative matrix and the Hessian matrix

Recall that $p_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}\vec{h}^T Hf(\vec{a})\vec{h}$,

where H is the Hessian and $\vec{h} = (\vec{x} - \vec{a})$

First, $f(\vec{a}) = e^0 \cdot \sin(0) = 0$

$$Df(x,y,z) = \begin{bmatrix} 2e^{2x-3y}\sin(5z) & -3e^{2x-3y}\sin(5z) & 5e^{2x-3y}\cos(5z) \end{bmatrix}$$

and so

$$Df(0,0,0) = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

Also,

$$Hf(x,y,z) = \begin{bmatrix} 4e^{2x-3y}\sin 5z & -6e^{2x-3y}\sin 5z & 10e^{2x-3y}\cos 5z \\ -6e^{2x-3y}\sin 5z & 9e^{2x-3y}\sin 5z & -15e^{2x-3y}\cos 5z \\ 10e^{2x-3y}\cos 5z & -15e^{2x-3y}\cos 5z & -25e^{2x-3y}\sin 5z \end{bmatrix}$$

$$\Rightarrow Hf(0,0,0) = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & -15 \\ 10 & -15 & 0 \end{bmatrix}.$$

Note that $\vec{h} = \vec{x} - \vec{0} = \vec{x}$

Date: November 13, 2019.

Putting it all together, we get

$$p_2(x) = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}\vec{x} + \frac{1}{2}\vec{x}^T \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & -15 \\ 10 & -15 & 0 \end{bmatrix}\vec{x}$$

(2) Identify and determine the nature of the critical point(s) of the function $f(x, y, z) = x^2 - xy + z^2 - 2xz + 6z$.

If f has a local extremum at $\vec{a}$, then $Df(\vec{a}) = \vec{0}$. So, we begin by finding $Df(x, y, z)$.

$$Df(x,y,z) = (2x - y - 2z \ , \ -x \ , \ 2z - 2x + 6)$$

From this we see that $-x = 0 \Rightarrow x = 0$.

Thus, $2z - 0 + 6 = 0 \Rightarrow z = -3$, and so $2(0) - y - 2(-3) = 0$
$\Rightarrow y = 6$.

There is one critical point at $\vec{a} = (0, 6, -3)$

To determine the nature of the critical point, we use the 2^{nd}-derivative test. So we find the Hessian of f evaluated at $\vec{a}$.

$$Hf(\vec{x}) = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

. Evaluating this at $\vec{a}$ we arrive at the same matrix.

The sequence of principal minors is $d_1 = 2$, $d_2 = -1$, $d_3 = -2$

Thus $\vec{a}$ is a saddle point.

(3) Use Lagrange multipliers to identify the critical point(s) of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the following constraint $x + y - z = 1$.

The theory of Lagrange multipliers says that a critical point occurs when

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

So we solve the following system:

$$\nabla f(\vec{x}) = (2x, 2y, 2z)$$

$$\nabla g(\vec{x}) = (1, 1, -1)$$

and so

$$2x = \lambda$$
$$2y = \lambda$$
$$2z = -\lambda$$
$$x + y - z = 1$$

We obtain that $\lambda = \frac{x}{2} = \frac{y}{2} = \frac{-z}{2} \Rightarrow x = y = -z$.

Substituting into the last equation, we obtain

$$x + x + x = 1 \Rightarrow x = \frac{1}{3} \Rightarrow y = \frac{1}{3}, z = \frac{-1}{3}.$$

Thus the only critical point is $\left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right)$.

(N.B. - The technique of determining the nature of the critical point you will know. However, given the timed environment of this exam, I chose not to ask you to apply it here.)