63 points total

MULTIVARIABLE CALCULUS
EXAM 2
FALL 2018

Name: Solution Key.

Honor Code Statement and Signature: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Calculators/notes/texts/cellphones are not permitted – the only permitted item is a writing utensil. Best of luck.

(1) (a) Write the definition of the *directional derivative of* f *at* $\mathbf{a}$ *in the direction of* $\mathbf{u}$, $D_{\mathbf{u}}f(\mathbf{a})$.

10 points

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$

for a scalar-valued function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\vec{a} \in X$.

(b) **Fill-in-the-blank** Theorem 6.2 (of Section 2.6) established that for differentiable f we have that $D_{\mathbf{u}}f(\mathbf{a})$ exists and can be computed as the dot product of the gradient of f at $\mathbf{a}$ and $\mathbf{u}$.

This is the conclusion of Theorem 6.2 of Section 6.2.

Note: Many students took this result as the definition wanted in part (a); this is incorrect to do.

(c) From a theorem of chapter 1, $D_{\mathbf{u}}f(\mathbf{a})$ can also be computed as $\|\nabla f(\mathbf{a})\|\|\mathbf{u}\|\cos(\Theta)$, where Θ is the measure of the angle between these two vectors. In terms of Θ, characterize when the directional derivative is maximized, minimized and zero.

Note that for $\vec{u}$ a unit vector $\|\vec{u}\| = 1$, and that for f and $\vec{a}$ fixed, $\|\nabla f(\vec{a})\|$ is a fixed, non-negative real number. Note also that $-1 \leq \cos\theta \leq 1$, and $\cos\theta = 1$ when $\theta = 0$, $\cos\theta = -1$ when $\theta = \pi$ and $\cos\theta = 0$ when $\theta = \frac{\pi}{2}$. These thus correspond to when the directional derivative is maximized, minimized and zero, respectively.

Date: November 15, 2018.

6 points (2) **Compute the path of steepest descent** Let $f(x,y) = xy$. Some level curves of f are sketched below. Find the curve traced by the path of steepest *descent* that goes through the point $(x,y) = (1,2)$.

An object moving in direction of steepest descent must have velocity vector parallel to $-\nabla f$. The curve y must satisfy

$$\frac{dy}{dx} = \frac{\partial f/\partial y}{\partial f/\partial x}$$ So we compute $\nabla f(x,y) = (y,x)$.

Thus $\frac{dy}{dx} = \frac{x}{y}$. This is a separable differential equation.

$$y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C.$$

We may find C as $(x,y) = (1,2)$ is a point on the curve.

$$\frac{2^2}{2} = \frac{1^2}{2} + C \Rightarrow C = \frac{3}{2}$$ Thus, $\frac{y^2}{2} = \frac{x^2}{2} + \frac{3}{2}$ is

the curve we seek. Note we only want the positive x and y that solve this as $x=1, y=2$ are both positive.

The figure on the following page comes from a Maple worksheet and gives a set of contour curves (with the values indicated in the command line) for the function $f(x,y) = xy$. Sketch the curve traced and indicate the direction of the path.

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> 
> with(plots):
> contourplot(x·y, x=-5..5, y=-5..5, contours = [-2,-1, 1, 2, 3, 4]);
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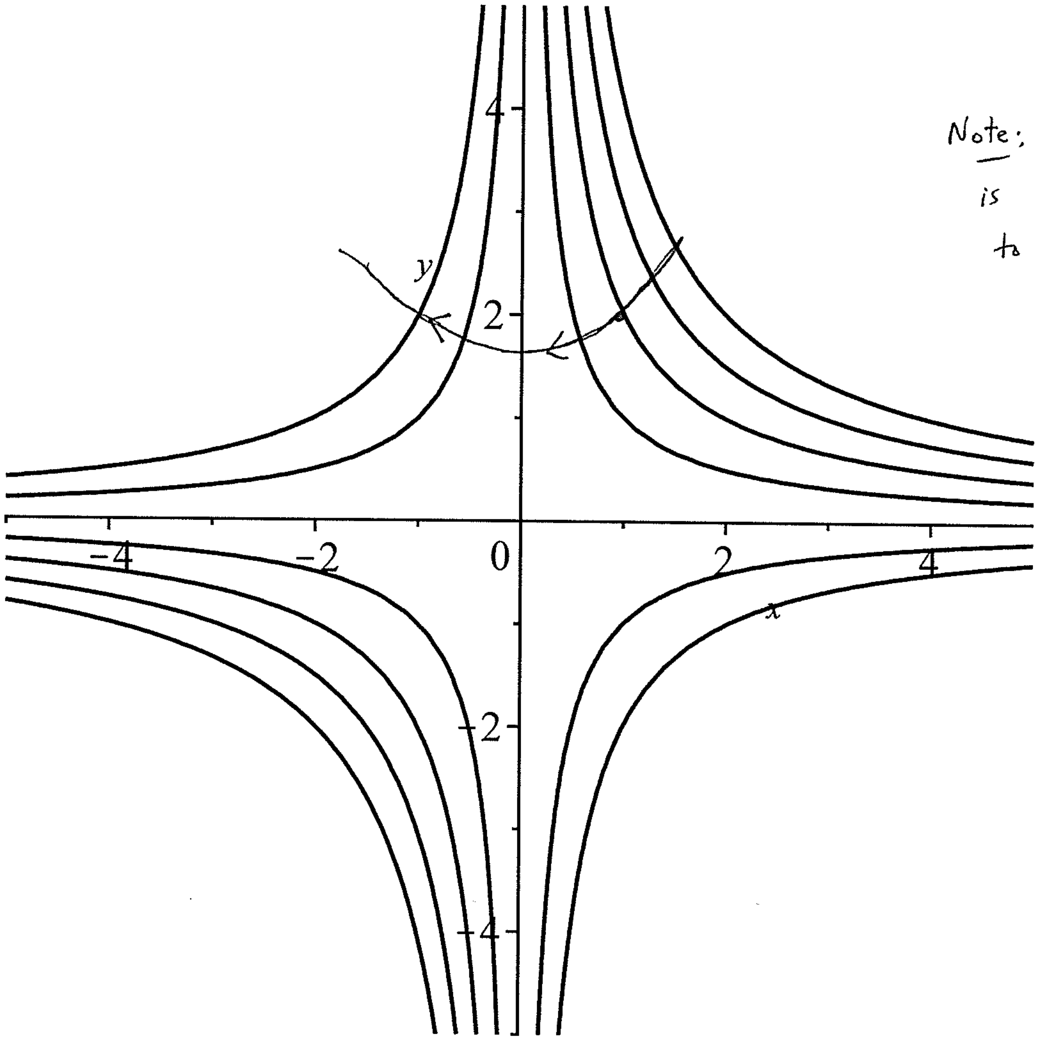

Note: the gradient is perpendicular to the level curves.

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> 
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15 points (3) Consider the following path,

$$\mathbf{x}(t) = (\cos(3t), \sin(3t), 2t^{3/2}).$$

(a) Find the velocity.

The velocity is

$$\vec{v}(t) = \vec{x}\,'(t) = \left(-3\sin 3t\ ,\ 3\cos 3t,\ 3\,t^{1/2}\right).$$

(b) Find the acceleration.

The acceleration is

$$\vec{a}(t) = \vec{x}\,''(t) = \left(-9\cos 3t\ ,-9\sin 3t\ ,\ \tfrac{3}{2}\,t^{-1/2}\right)$$

(c) Find the speed.

The speed is

$$\|\vec{v}(t)\| = \|\vec{x}\,'(t)\| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 9t}$$
$$= \sqrt{9 + 9t} = 3\sqrt{1+t}$$

(d) Give the unit tangent vector when $t = \pi$.

The unit tangent vector is

$$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{(-3\sin 3t,\ 3\cos 3t,\ 3t^{1/2})}{3\sqrt{1+t}}$$

At $t = \pi$ this is

$$\vec{T}(\pi) = \frac{(0\ ,\ -3\ ,\ 3\sqrt{\pi})}{3\sqrt{1+\pi}} = \left(0\ ,\ \frac{-1}{\sqrt{1+\pi}}\ ,\ \sqrt{\frac{\pi}{1+\pi}}\right)$$

(e) Find the length of the path between $t = 0$ and $t = 2$.

The length of the path is given by

$$L = \int_0^2 \|\vec{x}\,'(t)\|\, dt = \int_0^2 3\sqrt{1+t}\, dt .$$

This can be solved by making a u-substitution, $u = 1+t$, $du = 1dt$

Thus,

$$L = \frac{3(1+t)^{3/2}}{3/2}\Bigg|_0^2 = 2(1+t)^{3/2}\Big|_0^2 = 2(3)^{3/2} - 2(1)^{3/2}$$

$$= 2\left[3^{3/2} - 1\right].$$

6 points (4) Consider the vector field $\mathbf{F} = (y^3, y)$, which is sketched below. Consider the flow line through the point $(0, 5)$ (denote it $\mathbf{x}_1(t)$) and the flow line through the point $(0, -5)$ (denote it $\mathbf{x}_2(t)$).

(a) There is a qualitative difference between these two flow lines. What is it?

By "reading" the vector field, we see that $\vec{x}_1(t)$ "flows" from left to right whereas $\vec{x}_2(t)$ "flows" from right to left.

(b) What is the divergence of $\mathbf{F}$ at the point $(0, -5)$?

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial (y^3)}{\partial x} + \frac{\partial (y)}{\partial y} = 0 + 1 = 1$$

So the divergence is 1 everywhere, including at $(0,-5)$.

(c) Does the divergence of $\mathbf{F}$ along the path $\mathbf{x}_2(t)$ increase, decrease or remain constant as t grows? Justify.

By part (b) it is constant.

8 points. (5) Determine the first-order and second-order Taylor polynomial of $f(x,y) = e^{2x}\cos(y)$ at the point $(0,\pi/2)$.

To use Taylor's Theorem, we need to make the following calculations:

$$f(0,\pi/2) = e^0 \cdot \cos\left(\frac{\pi}{2}\right) = 0$$

$f_x = 2e^{2x}\cos y$ and $f_x(0,\pi/2) = 2e^0 \cdot \cos\frac{\pi}{2} = 0$

$f_y = -e^{2x}\sin(y)$ $\quad f_y(0,\pi/2) = -e^0 \cdot \sin\frac{\pi}{2} = -1$

Thus, $P_1(\vec{x}) = 0 + 0(x-0) + -1\,(y - \pi/2) = -y + \pi/2$

Continuing to the second-order Taylor polynomial

$f_{xx} = 4e^{2x}\cos y$ $\quad f_{xx}(0,\pi/2) = 4e^0 \cdot \cos\frac{\pi}{2} = 0$

$f_{xy} = -2e^{2x}\sin y$ and so $f_{xy}(0,\pi/2) = -2e^0 \cdot \sin\frac{\pi}{2} = -2$

$f_{yy} = -e^{2x}\cos y$ $\quad f_{yy}(0,\pi/2) = -e^0 \cdot \cos\frac{\pi}{2} = 0$

Thus $P_2(\vec{x}) = -y + \frac{\pi}{2} + \frac{1}{2}\cdot 0\,(x-0)^2 + -2\,(x-0)\left(y-\frac{\pi}{2}\right) + \frac{1}{2}\cdot 0 \cdot \left(y-\frac{\pi}{2}\right)$

$$= -y + \frac{\pi}{2} - 2x\left(y - \frac{\pi}{2}\right) = -y + \frac{\pi}{2} - 2xy + x\pi$$

10 points. (6) Find the (absolute) maximum and minimum values of $f(x, y) = \sin(x)\cos(y)$ on the square $R = \{(x, y) | 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi\}$.

We begin by finding critical points of $f(x,y)$.

So we set $Df(x,y)$ to $\vec{0}$ and solve.

$$Df(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y \end{bmatrix}.$$

$\cos x \cos y = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $y = \frac{\pi}{2}, \frac{3\pi}{2}$

$-\sin x \sin y = 0$ when $x = 0, \pi, 2\pi$ or $y = 0, \pi, 2\pi$.

We need to satisfy these two equations simultaneously, which means critical points are: $(\frac{\pi}{2}, 0), (\frac{\pi}{2}, \pi), (\frac{\pi}{2}, 2\pi)$

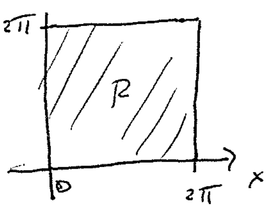
$(\frac{3\pi}{2}, 0), (\frac{3\pi}{2}, \pi), (\frac{3\pi}{2}, 2\pi)$

$(0, \frac{\pi}{2}), (0, \frac{3\pi}{2})$

$(\pi, \frac{\pi}{2}), (\pi, \frac{3\pi}{2})$

$(2\pi, \frac{\pi}{2}) (2\pi, \frac{3\pi}{2})$

Note that



has 4 edges. ① Along the bottom edge, $y=0$ and $f(x,y) = \sin x$ which has a max value at $x = \frac{\pi}{2}$ of 1, and a min value at $x = \frac{3\pi}{2}$ of -1.

② along the top edge, $y = 2\pi$ and $f(x,y) = \sin x$ which has a max value at $x = \frac{\pi}{2}$ of 1 and a min value at $x = \frac{3\pi}{2}$ of -1

③ Along the left edge $x = 0$ and $f(x,y) = \cancel{\cos y}\ 0$ ~~which has a max value at $y=0$ and $y=2\pi$ of 1, and a min value of -1 at $y=\pi$~~

④ Along the right edge $x = 2\pi$ and $f'(x,y) = 0$

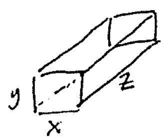
Let us create a table of critical points and corresponding out-put values. From these we can identify which are maximums and minimums.

critical point (x,y)	$f(x,y)$	nature
$(\pi/2, 0)$	1	maximum
$(\pi/2, \pi)$	-1	minimum
$(\pi/2, 2\pi)$	1	maximum
$(3\pi/2, 0)$	-1	minimum
$(3\pi/2, \pi)$	1	maximum
$(3\pi/2, 2\pi)$	-1	minimum.
$(0, \pi/2)$	0	
$(0, 3\pi/2)$	0	
$(\pi, \pi/2)$	0	
$(\pi, 3\pi/2)$	0	
$(2\pi, \pi/2)$	0	
$(2\pi, 3\pi/2)$	0	

8 points

(7) Let the dimensions of a box (i.e. a parallelepiped) be x inches, by y inches, by z inches. Use the method of Lagrange multipliers to determine the dimensions of the largest box if the sum of the length and the girth (i.e. the perimeter of the cross section perpendicular to the length axis - i.e. think of your waist size) can be at most 108 inches.

Let's sketch a box of these dimensions:



From this we see that the volume is given by $V = xyz$, which is our objective function (i.e. the function we seek to maximize). There is a constraint given: $z + 2x + 2y \le 108$. $\left(g(x,y,z) \le 108\right)$

Applying the method of Lagrange multipliers, we must solve:

$$\nabla V(x,y,z) = \lambda \nabla g(x,y,z)$$

$$g(x,y,z) = 108$$

This is:

$$yz = 2\lambda \quad ①$$

$$xz = 2\lambda \quad ②$$

$$xy = 1\lambda \quad ③$$

$$z + 2x + 2y = 108 \quad ④$$

From the first two equations we obtain $xz = yz \Leftrightarrow z(x-y) = 0$

We reject $z = 0$ as for then $V = 0$. Thus $x = y$. Thus ③ becomes $x^2 = \lambda$

$\Leftrightarrow 2x^2 = 2\lambda$. By ② $2x^2 = xz \Rightarrow x = \frac{z}{2}$. Substituting into ④

we get $2x + 2x + 2x = 108 \Rightarrow x = \frac{108}{6} = 18 \Rightarrow y = 18 \Rightarrow z = 36$.

This corresponds to dimensions w/ max volume and meeting the constraint.