MULTIVARIABLE CALCULUS
EXAM 2
FALL ~~2013~~ 2014

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Each problem is worth 10 points. Calculators/notes/texts/cell-phones are not permitted – the only permitted item is a writing utensil. Best of luck.

(1) Calculate the velocity, speed, and acceleration of the path $\mathbf{x}(t) = (t, t^2, t^3)$. Also, sketch an image of the path, using arrows to indicate the direction in which the parameter increases. (If it is difficult to sketch the path, consider sketching its projections onto the 3 coordinate planes.)

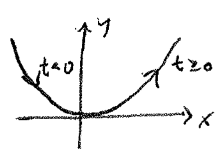
The velocity is $\vec{v}(t) = \vec{x}'(t) = (1, 2t, 3t^2)$.

The speed is the norm of the velocity, $\|\vec{v}(t)\| = \sqrt{1+4t^2+9t^4}$

The acceleration is $\vec{a}(t) = \vec{v}'(t) = (0, 2, 6t)$.

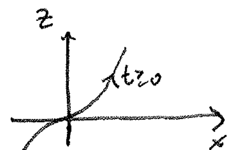
Let's follow the hint and begin our sketch by first sketching projections onto the 3 coordinate planes.

For the x,y-plane: as $x=t$ and $y=t^2$, we may eliminate the parameter t to obtain $y=x^2$:



Arrow indicates direction of increasing t.

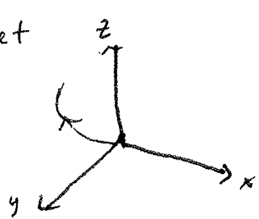
For the x,z-plane: as $x=t$ and $z=t^3$, we may eliminate the parameter t to obtain $z=x^3$:

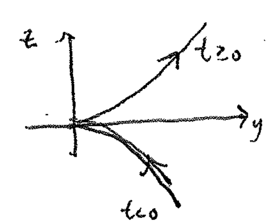


For the y,z-plane: as $y=t^2$, $z=t^3$, we may eliminate the parameter t to obtain $z=\pm y^{3/2}$

Date: November ~~14, 2013~~ 6 2014.

Putting these together, we get (for $t\geq 0$) a spiral-looking curve





(2) Find the arc length parameter $s = s(t)$ for the path

$$\mathbf{x}(t) = e^{at}\cos bt\ \mathbf{i} + e^{at}\sin bt\ \mathbf{j} + e^{at}\ \mathbf{k}.$$

Also, express the original parameter t in terms of s and, thereby, reparametrize $\mathbf{x}$ in terms of s.

Recall that the length parameter is given by

$$s(t) = \int_0^t \|\vec{x}'(\tau)\|\, d\tau.$$

First, $\vec{x}'(\tau) = \left(-be^{a\tau}\sin b\tau + ae^{a\tau}\cos b\tau,\ be^{a\tau}\cos b\tau + ae^{a\tau}\sin b\tau,\ ae^{a\tau}\right)$

Thus, $\|\vec{x}'(\tau)\| = \sqrt{a^2e^{2a\tau}\cos^2 b\tau \cancel{-2abe^{2a\tau}\sin b\tau\cos b\tau} + b^2e^{2a\tau}\sin^2 b\tau + b^2e^{2a\tau}\cos^2 b\tau \cancel{+2abe^{2a\tau}\sin b\tau\cos b\tau} + a^2e^{2a\tau}\sin^2 b\tau + a^2e^{2a\tau}}$

$$= \sqrt{a^2e^{2a\tau} + a^2e^{2a\tau} + b^2e^{2a\tau}}$$

$$= e^{a\tau}\sqrt{2a^2+b^2}$$

Thus, $s(t) = \int_0^t e^{a\tau}\sqrt{2a^2+b^2}\, d\tau = \frac{1}{a}\sqrt{2a^2+b^2}\, e^{a\tau}\Big|_0^t = \frac{\sqrt{2a^2+b^2}}{a}e^{at} - \frac{\sqrt{2a^2+b^2}}{a}\cdot 1$

$$= \frac{\sqrt{2a^2+b^2}}{a}\left(e^{at}-1\right).$$

Thus, $s = \frac{\sqrt{2a^2+b^2}}{a}\left(e^{at}-1\right)$.

To continue, we solve for t in terms of s: $\frac{as}{\sqrt{2a^2+b^2}} = e^{at} - 1$

$$\Rightarrow \frac{as}{\sqrt{2a^2+b^2}} + 1 = e^{at} \Rightarrow \frac{\ln\left(\frac{as}{\sqrt{2a^2+b^2}}+1\right)}{a} = t$$

(3) Calculate the flow line $\mathbf{x}(t)$ of the vector field $\mathbf{F}(x, y, z) = 2\mathbf{i} - 3y\mathbf{j} + z^3\mathbf{k}$ at the point $\mathbf{x}(0) = (3, 5, 7)$.

According to the definition of flow line: $\vec{x}(t)$ is a differentiable path such that $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$

So $\vec{x}\,'(t) = (2, -3y, z^3)$

Thus, we have 3 differentiable equations to solve - each is separable.

(1) $\frac{dx}{dt} = 2 \Rightarrow \int dx = \int 2\,dt \Rightarrow x = 2t + C$

As $x(0) = 3$, we have $x = 2t + 3$.

(2) $\frac{dy}{dt} = -3y \Rightarrow \int \frac{dy}{y} = \int -3\,dt \Rightarrow \ln|y| = -3t + C$

$\Rightarrow y = e^{-3t+C} = Ce^{-3t}$

As $y(0) = 5$, we have $5 = Ce^0 \Rightarrow C = 5$.

$\Rightarrow y = 5e^{-3t}$

(3) $\frac{dz}{dt} = z^3 \Rightarrow \int z^{-3}\,dz = \int dt \Rightarrow -\frac{1}{2}z^{-2} = t + C$

$\Rightarrow \frac{1}{z^2} = -2t + C.$

As $z(0) = 7$, $\frac{1}{49} = -2(0) + C \Rightarrow C = \frac{1}{49}$.

$\Rightarrow \frac{1}{z^2} = -2t + \frac{1}{49}$

$z^2 = \frac{1}{-2t + 1/49} = \frac{49}{-98t + 1}$

$z = \frac{7}{\sqrt{1 - 98t}}$

Thus,

$\vec{x}(t) = \left(2t + 3,\ 5e^{-3t},\ \frac{7}{\sqrt{1 - 98t}}\right)$

(4) Establish the given identity.

$$\nabla \times (f\mathbf{F}) = f\ \nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$$

$$\nabla \times (f\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix}$$

by definition of curl.

$$= \left(\frac{\partial fF_3}{\partial y} - \frac{\partial fF_2}{\partial z} \right)\vec{i} - \left(\frac{\partial fF_3}{\partial x} - \frac{\partial fF_1}{\partial z} \right)\vec{j} + \left(\frac{\partial fF_2}{\partial x} - \frac{\partial fF_1}{\partial y} \right)\vec{k}$$

by computing the given determinant

$$= \left(\frac{\partial f}{\partial y} F_3 + \frac{\partial F_3}{\partial y} f - \frac{\partial f}{\partial z} F_2 - \frac{\partial F_2}{\partial z} f \right)\vec{i}$$

$$- \left(\frac{\partial f}{\partial x} F_3 + \frac{\partial F_3}{\partial x} f - \frac{\partial f}{\partial z} F_1 - \frac{\partial F_1}{\partial z} f \right)\vec{j}$$

by the product rule!!

$$+ \left(\frac{\partial f}{\partial x} F_2 + \frac{\partial F_2}{\partial x} f - \frac{\partial f}{\partial y} F_1 - \frac{\partial F_1}{\partial y} f \right)\vec{k}$$

$$= f\ \nabla \times \vec{F} + \nabla f \times \vec{F}$$

(5) Find the first- and second-order Taylor polynomials for the given function f at the point $\mathbf{a} = (0,0,0)$.

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2+1}$$

To do this, we need to find the value of the function at $\vec{a}$, and the value of all partial derivatives at $\vec{a}$, and value of all second-order partial derivatives at $\vec{a}$.

So, $f(\vec{a}) = \dfrac{1}{0^2+0^2+0^2+1} = 1$

$\dfrac{\partial f}{\partial x} = \dfrac{-2x}{(x^2+y^2+z^2+1)^2}$ (by the quotient rule) and so $\dfrac{\partial f(\vec{a})}{\partial x} = \dfrac{0}{1}$

Similar calculations follow for $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, and so $\dfrac{\partial f(\vec{a})}{\partial y} = \dfrac{\partial f(\vec{a})}{\partial z} = 0$

Thus, the first-order Taylor polynomial is $P_1(\vec{x}) = 1$

Finding the second-order partials:

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{-2(x^2+y^2+z^2+1)^2 + 2x \cdot 2(x^2+y^2+z^2+1)\cdot 2x}{(x^2+y^2+z^2+1)^4} = \frac{-2x^2-2y^2-2z^2-2+8x^2}{(x^2+y^2+z^2+1)^3}$$

$$= \frac{6x^2-2y^2-2z^2-2}{(x^2+y^2+z^2+1)^3}$$

and so, $\dfrac{\partial^2 f}{\partial x \partial x}(\vec{a}) = \dfrac{-2}{1}$. Similar calculations show $\dfrac{\partial^2 f(\vec{a})}{\partial y \partial y} = \dfrac{\partial^2 f(\vec{a})}{\partial z \partial z} = -2$

The mixed partials:

$$\frac{\partial f}{\partial y \partial x} = \frac{0 + 2x\cdot 2(x^2+y^2+z^2+1)\cdot 2y}{(x^2+y^2+z^2+1)^4} = \frac{8xy}{(x^2+y^2+z^2+1)} \Rightarrow \frac{\partial f}{\partial y \partial x}(\vec{a}) = 0$$

and like-wise for others.

Thus, $$P_2(\vec{x}) = 1 + 0 + \frac{1}{2}\begin{bmatrix}(x-0) & (y-0) & (z-0)\end{bmatrix}\begin{bmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = 1 + \frac{1}{2}(-2x^2-2y^2-2z^2)$$

$$= \frac{1}{2} - x^2 - y^2 - z^2$$

(6) Identify and determine the nature of the critical points of the following function,
$$f(x,y) = e^{-y}(x^2 - y^2).$$

Extrema only occur where $Df(\vec{a}) = \vec{0}$. So,

$$Df = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 2e^{-y}x & e^{-y}(-2y) - e^{-y}(x^2-y^2) \end{bmatrix} = \begin{bmatrix} 2e^{-y}x & -e^{-y}(x^2-y^2+2y) \end{bmatrix}$$

When does this equal $[0,0]$? Well, if $2e^{-y}x = 0$, we must have $x=0$. From the $f_y = 0$ and $x=0$, we get $0 = -e^{-y}(x^2-y^2+2y)$

(Note $e^{-y} > 0$) $\Rightarrow 0 = e^{-y}(-y^2+2y)$

$\Rightarrow 0 = -y^2 + 2y$

$0 = y(-y+2)$

$\Rightarrow y = 0$ or $y = 2$.

Thus, critical points at $x=0, y=0$ and $x=0, y=2$.

We test each to determine the nature using the second derivative test:

We form the Hessian matrix and find the sequence of principal minors.

$$Hf = \begin{bmatrix} 2e^{-y} & -2e^{-y}x \\ -2e^{-y}x & -e^{-y}(-2y+2) + (x^2-y^2+2y)(-e^{-y}) \end{bmatrix} \Rightarrow Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$d_1 = 2$, $d_2 = -8$ $\Rightarrow (0,0)$ is a saddle point

$$\Rightarrow Hf(0,2) = \begin{bmatrix} 2e^{-2} & 0 \\ 0 & 2e^{-2} \end{bmatrix}$$

$d_1 > 0$, $d_2 > 0$ $\Rightarrow (0,2)$ is a local minimum.

(7) Find the points on the ellipse $3x^2 - 4xy + 3y^2 = 50$ that are nearest and farthest from the origin. (Hint: one can avoid use of the second derivative test for constrained local extrema. How? Regardless, I've attached a photocopy of page 288 of the text.)

We seek to optimize the distance subject to the constraint that the point is on the ellipse. Instead of optimizing the distance function, which involves a square-root, we'll optimize w.r.t. the square of the distance: $f(x,y) = x^2 + y^2$.

We use the method of Lagrange multipliers: we form the vector equation $\nabla f(\vec{x}) = \lambda \nabla g(\vec{x})$, where $g(x,y) = 3x^2 - 4xy + 3y^2$, and solve the system $\nabla f(\vec{x}) = \lambda \nabla g(x)$
$g(\vec{x}) = c$

$\nabla f(x,y) = (2x \quad 2y)$
$\nabla g(x,y) = (6x - 4y \quad -4x + 6y)$
$\Rightarrow$
$2x = \lambda(6x - 4y)$ ①
$2y = \lambda(-4x + 6y)$ ②
$3x^2 - 4xy + 3y^2 = 50$ ③

① and ② give $\lambda = \frac{2x}{6x-4y} = \frac{2y}{6y-4x} \Rightarrow 12xy - 8x^2 = 12xy - 8y^2$
$\Rightarrow x^2 = y^2.$
$\Rightarrow y = \pm x.$

By substitution: if $y = x$, then ③ is $3x^2 - 4x^2 + 3x^2 = 50$
$\Rightarrow 2x^2 = 50 \quad x = \pm 5.$
and so $(5,5)$ and $(-5,-5)$ are critical points.

; if $y = -x$, then ③ is $3x^2 + 4x^2 + 3x^2 = 50$
$x^2 = 5 \Rightarrow x = \pm\sqrt{5}$
and so $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$ are critical points

We use Extreme Value Theorem:
As $f(5,5) = f(-5,-5) = 50$, and $f(-\sqrt{5}, \sqrt{5}) = f(\sqrt{5}, -\sqrt{5}) = 10$, the first two correspond to max and the other to

Second derivative test for constrained local extrema. Given a constrained critical point $\mathbf{a}$ of f subject to the conditions $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2, \ldots, g_k(\mathbf{x}) = c_k$, consider the matrix

$$HL(\lambda; \mathbf{a}) = \begin{bmatrix} 0 & \cdots & 0 & -\frac{\partial g_1}{\partial x_1}(\mathbf{a}) & \cdots & -\frac{\partial g_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{\partial g_k}{\partial x_1}(\mathbf{a}) & \cdots & -\frac{\partial g_k}{\partial x_n}(\mathbf{a}) \\ -\frac{\partial g_1}{\partial x_1}(\mathbf{a}) & \cdots & -\frac{\partial g_k}{\partial x_1}(\mathbf{a}) & h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_1}{\partial x_n}(\mathbf{a}) & \cdots & -\frac{\partial g_k}{\partial x_n}(\mathbf{a}) & h_{n1} & \cdots & h_{nn} \end{bmatrix},$$

where

$$h_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}) - \lambda_1 \frac{\partial^2 g_1}{\partial x_j \partial x_i}(\mathbf{a}) - \lambda_2 \frac{\partial^2 g_2}{\partial x_j \partial x_i}(\mathbf{a}) - \cdots - \lambda_k \frac{\partial^2 g_k}{\partial x_j \partial x_i}(\mathbf{a}).$$

(Note that $HL(\lambda; \mathbf{a})$ is an $(n+k) \times (n+k)$ matrix.) By relabeling the variables as necessary, assume that

$$\det \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial g_1}{\partial x_k}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial g_k}{\partial x_k}(\mathbf{a}) \end{bmatrix} \neq 0.$$

As in the unconstrained case, let H_j be the upper leftmost $j \times j$ submatrix of $HL(\lambda, \mathbf{a})$. For $j = 1, 2, \ldots, k+n$, let $d_j = \det H_j$, and calculate the following sequence of $n - k$ numbers:

$$(-1)^k d_{2k+1}, \quad (-1)^k d_{2k+2}, \ldots, \quad (-1)^k d_{k+n}. \tag{1}$$

Note that, if $k \geq 1$, the sequence in (1) is *not* the complete sequence of principal minors of $HL(\lambda, \mathbf{a})$. Assume $d_{k+n} = \det HL(\lambda, \mathbf{a}) \neq 0$. The numerical test is as follows:

1. If the sequence in (1) consists entirely of positive numbers, then f has a local minimum at $\mathbf{a}$ subject to the constraints
$$g_1(\mathbf{x}) = c_1, \quad g_2(\mathbf{x}) = c_2, \ldots, \quad g_k(\mathbf{x}) = c_k.$$
2. If the sequence in (1) begins with a negative number and thereafter alternates in sign, then f has a local maximum at $\mathbf{a}$ subject to the constraints
$$g_1(\mathbf{x}) = c_1, \quad g_2(\mathbf{x}) = c_2, \ldots, \quad g_k(\mathbf{x}) = c_k.$$
3. If neither case 1 nor case 2 holds, then f has a constrained saddle point at $\mathbf{a}$.

In the event that $\det HL(\lambda, \mathbf{a}) = 0$, the constrained critical point $\mathbf{a}$ is **degenerate**, and we must use another method to determine whether or not it is the site of an extremum.