65 points

Average score: $\frac{53.2}{65}$

MULTIVARIABLE CALCULUS
EXAM 2
FALL 2013

Name: Answer Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam ~~—~~.

Directions: Complete all problems. Justify all answers/solutions. Definitions are worth 4 points, true/false questions are worth 2 points each, and any other problem is worth 10 points. Calculators are not permitted. Best of luck.

(1) [5 points] Find the value of the directional derivative of $f(x, y, z) = 3x^2y^2z^2 + 2xyz + z$ at $\mathbf{a} = (4, 5, 6)$ in the direction of the vector $(1, 1, 2)$.

We begin by normalizing the vector $(1,1,2)$ and obtain $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$
We use Theorem 6.2 of Section 2.6, which states that for for f differentiable at $\vec{a}$, $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$. So, we begin by finding the gradient:

$$\nabla f = \left(6xy^2z^2 + 2yz,\ 6x^2yz^2 + 2xz,\ 6x^2y^2z + 2xy + 1\right).$$

$$\nabla f(\vec{a}) = \left(6\cdot 4\cdot 5^2\cdot 6^2 + 2\cdot 5\cdot 6,\ 6\cdot 4^2\cdot 5\cdot 6^2 + 2\cdot 5\cdot 6,\ 6\cdot 4^2\cdot 5^2\cdot 6 + 2\cdot 4\cdot 5 + 1\right)$$

who decided on this problem?!

And so, $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

(2) [5 points] Calculate the velocity, speed, and acceleration of the following path: $\mathbf{x}(t) = (5\sin t, 3\cos t)$.

The velocity is $\vec{x}'(t) = (5\cos t, -3\sin t)$

The speed is $\|\vec{x}'(t)\| = \sqrt{25\cos^2 t + 9\sin^2 t} = \sqrt{9 + 16\cos^2 t}$

The acceleration is $\vec{x}''(t) = (-5\sin t, -3\cos t)$

Date: November 14, 2013.

(3) [5 points] Calculate the length of the path $\mathbf{x}(t) = (t^3, 3t^2, 6t)$ for $-1 \le t \le 2$.

The length of a C^1 path is found by integrating its speed:

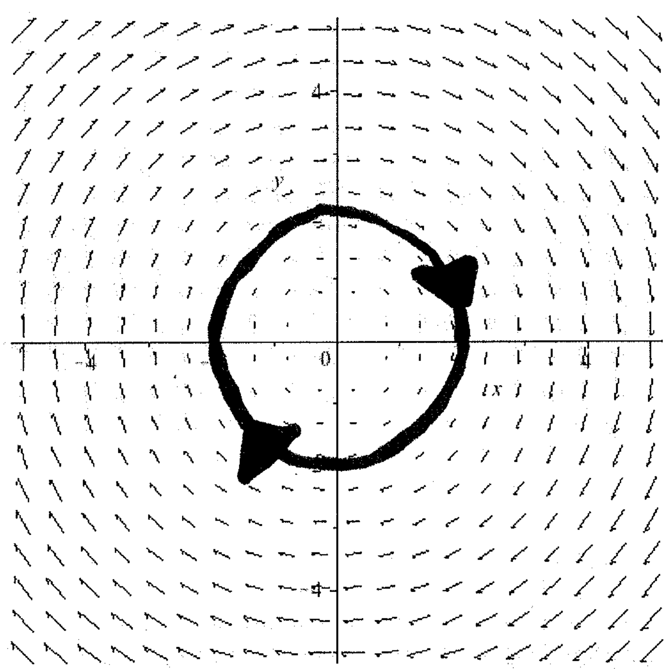
$$L(\vec{x}) = \int_{-1}^{2} \sqrt{9t^4 + 36t^2 + 36}\, dt = 3\int_{-1}^{2} \sqrt{t^4 + 4t^2 + 4}\, dt$$

$$= 3\int_{-1}^{2} \sqrt{(t^2+2)^2}\, dt = 3\int_{-1}^{2} t^2 + 2\, dt$$

$$= t^3 + 6t \Big|_{-1}^{2} = 20 - (-7)$$

$$= 27$$

(4) [5 points] The following figure is the vector field for the function $\mathbf{F}(x, y) = (y, -x)$. Use the figure to sketch the flow line that contains the point $(2, 0)$.



(5) [5 points] Is it possible for the vector field of the previous problem to also be a gradient field? If yes, then find a scalar-valued function f that would produce this $\mathbf{F}$. If no, explain why.

Recall that a gradient field on $\mathbb{R}^2$ is a vector field $\vec{F}: X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ such that $\vec{F}(\vec{x}) = \nabla f(\vec{x})$, where f is a scalar valued function.

So we wonder if there exists an $f: X \to \mathbb{R}$ such that

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, -x).$$

If $\frac{\partial f}{\partial x} = y$, then $f = xy + C_1$. If $\frac{\partial f}{\partial y} = -x$, then $f = -xy + C_2$

And we see that it is impossible to satisfy these both simultaneously.

(6) [5 points] Why would asking you to find the curl of $\mathbf{F}$, where $\mathbf{F}$ is as given in problem 4, be a trick question?

The curl is defined on functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ only. The $\vec{F}$ of problem (4) is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(7) [10 points] Find the first- and second-order Taylor polynomials for the function $f(x, y) = xy + \sin x \cos y$ at $\mathbf{a} = (\pi, \pi)$.

We begin by finding all first and second partial derivatives.

$f_x = y + \cos x \cos y$ $\qquad$ $f_{xx} = -\sin x \cos y$

$f_y = x - \sin x \sin y$ $\qquad$ $f_{yy} = -\sin x \cos y$

$f_{xy} = f_{yx} = 1 - \cos x \sin y$

Evaluate each at $\vec{a} = (\pi, \pi)$

$f_x(\pi,\pi) = \pi + 1$, $f_y(\pi,\pi) = \pi$, $f_{xx}(\pi,\pi) = 0$, $f_{yy}(\pi,\pi) = 0$, $f_{xy}(\pi,\pi) = 1$

The first-order Taylor polynomial is

$$P_1(\vec{x}) = \pi^2 + (\pi + 1)(x - \pi) + \pi(y - \pi)$$

$$= \pi^2 + (\pi+1)x - \pi^2 - \pi + \pi y - \pi^2$$

$$= -\pi^2 - \pi + (\pi + 1)x + \pi y$$

The second-order Taylor polynomial is

$$P_2(\vec{x}) = \pi^2 + (\pi+1)(x-\pi) + \pi(y-\pi) + 0 \cdot \tfrac{1}{2}(x-\pi)^2 + 0 \cdot \tfrac{1}{2}(y-\pi)^2$$
$$+ (x-\pi)(y-\pi)$$

$$= -\pi^2 - \pi + (\pi+1)x + \pi y + xy - y\pi - x\pi + \pi^2$$

$$= -\pi + x + xy$$

(8) [10 points] Identify and determine the nature of the two critical points of the function $f(x, y, z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$.

We find the critical points by setting Df to $\vec{0}$.

$$f_x = 3x^2 + z^2 - 6x$$

$$f_y = 2y$$

$$f_z = 2xz + 4z = 2z(x+2)$$

Setting f_y equal to zero, we see $y=0$. Setting f_z equal to zero, we see $z=0$ or $x=-2$. If $z=0$, then $0 = 3x^2 - 6x = 3x(x-2)$ and so $x=0$ or $x=2$. If $x=-2$, then $0 = 3(-2)^2 + z^2 + 12 \Rightarrow z^2 = -24$, and this has no solution in the reals. Thus, the critical points are $(0,0,0)$ and $(2,0,0)$.

To determine the nature of these critical points, we first find the Hessian.

$$Hf = \begin{bmatrix} 6x-6 & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4 \end{bmatrix}$$

We calculate the sequence of principal minors for each critical point:

$$d_1 = 6x - 6$$

$$d_2 = 12x - 12$$

$$d_3 = 2\left[(6x-6)(2x+4) - 4z^2\right]$$

$d_1(0,0,0) = -6$

$d_2(0,0,0) = -12$

$d_3(0,0,0) = -48$

Thus, $(0,0,0)$ is a saddle point.

$d_1(2,0,0) = 6$

$d_2(2,0,0) = 12$

$d_3(2,0,0) = 96$

Thus, $(2,0,0)$ is a local minimum

(9) [10 points] A farmer has determined that her cornfield will yield corn (in bushels) according to the formula

$$B(x, y) = 4x^2 + y^2 + 600,$$

where x denotes the amount of water (measured in hundreds of gallons) used to irrigate the field and y the number of pounds of fertilizer applied to the field. The fertilizer costs \$10 per pound and water costs \$15 per hundred gallons (wow, expensive water!). If she can allot \$500 to prepare her field through irrigation and fertilization, use a Lagrange multiplier to determine how much water and fertilizer she should purchase in order to maximize her yield.

$C(x,y)$ is a cost function: $C(x,y) = 15x + 10y$.

And so the constraint is $15x + 10y = 500$

B and C are of class C^1, and so the theorem of Lagrange applies.

We will solve the system $\nabla B(\vec{x}) = \lambda \nabla C(x)$, $C(\vec{x}) = 500$.

$\nabla B(x,y) = (8x, 2y)$

$\nabla C(x,y) = (15, 10)$

$$\Rightarrow \left.\begin{array}{l} 8x = 15\lambda \\ 2y = 10\lambda \end{array}\right\} \Rightarrow 8x = 3y \Rightarrow y = \frac{8}{3}x$$

$15x + 10y = 500$

$$\Rightarrow 15x + 10 \cdot \frac{8}{3}x = 500 \Rightarrow \frac{125}{3}x = 500 \Rightarrow 125x = 1500$$

$x = 12$

Thus, $15 \cdot 12 + 10y = 500 \Rightarrow 180 + 10y = 500 \Rightarrow y = 32$.

Thus $(12, 32)$ is a critical point. But is it a maximum on $\{(x,y) \mid 0 \le x \le \frac{500}{15}, 0 \le y \le 50\}$?

$B(12, 32) = 4 \cdot 144 + 1024 + 600 = 2200$. But $B(\frac{500}{15}, 0) = 5044.\bar{4}$ and $B(0, 50) = 3{,}100$. So, she should spend her \$500 on water only.

(10) [5 points] Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Verify that $\nabla||\mathbf{r}||^n = n||\mathbf{r}||^{n-2}\mathbf{r}$.

By definition of the norm $||\vec{r}|| = (x^2+y^2+z^2)^{1/2}$.

Thus, $||\vec{r}||^n = (x^2+y^2+z^2)^{n/2}$.

This is a scalar function and the del operator acts on it in the following way:

$$\nabla||\vec{r}||^n = \nabla(x^2+y^2+z^2)^{n/2} = \left(\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2x, \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2y, \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2z\right)$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}}(2x, 2y, 2z)$$

$$= n\left((x^2+y^2+z^2)^{1/2}\right)^{n-2}(x, y, z)$$

$$= n\,||\vec{r}||^{n-2}\,\vec{r}$$

$\square$