MULTIVARIABLE CALCULUS
EXAM 1
SPRING 2024

Name:
Honor Code Statement:

Directions: You may OMIT one **5-point** question; indicate which you omit by putting a slash through it. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. A formula sheet is on the last page. Best of luck!

(1) [5 points] Where does the line $x = 1 - 4t, y = t - 3/2, z = 2t + 1$ intersect the plane $5x - 2y + z = 1$?

Date: π day, 2024.

(2) [5 points] Show that the vectors $||\mathbf{b}||\mathbf{a}+||\mathbf{a}||\mathbf{b}$ and $||\mathbf{b}||\mathbf{a}-||\mathbf{a}||\mathbf{b}$ are orthogonal where $\mathbf{a}=(3,4,12)$ and $\mathbf{b}=(1,0,1)$.

(3) [10 points] The volume of a parallelepiped is given by two formulas:

$$||\mathbf{a} \times \mathbf{b}||||\mathbf{c}|||\cos(\theta)|$$

and

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.$$

(a) Compute the volume of the parallelepiped when $\mathbf{a} = (1, 0, 1), \mathbf{b} = (1, 1, 1)$ and $\mathbf{c} = (0, 0, 1)$.

(b) What is θ referring to in the formula?

(c) What are the set of possible values of θ?

(d) For which values of θ can the absolute value sign be “dropped”?

(e) If the absolute valued sign was “dropped” for a value of θ for which it shouldn’t, what non-sensical result would be obtained?

(4) [5 points] Find the values for A so that the planes

$$Ax - y + z = 1$$

and

$$3Ax + Ay - 2z = 5$$

are perpendicular.

(5) [5 points] It is true that for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, we have

$$||\mathbf{a} - \mathbf{b}|| \leq ||\mathbf{a} - \mathbf{c}|| + ||\mathbf{c} - \mathbf{b}||.$$

Illustrate this inequality. That is, draw a picture that highlights the important aspects of the inequality. If you were to name this inequality, what name would you give it?

(6) [10 points] Consider the following equation:

$$x^2 + y^2 + \frac{z^2}{4} = 1.$$

(a) Sketch several level curves of this surface.

(b) Sketch sections of the surface by planes of the form $x = c$.

(c) Sketch sections of the surface by planes of the form $y = c$.

(d) Describe the surface or make a sketch of it.

(7) [5 points] Show that the following limit does not exist by computing the limits along the lines $y = x$ and another line of your choosing.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

(8) [10 points]. Given $f(x, y, z) = \sin(xyz)$ and the point $\mathbf{a} = (\pi, 0, \pi/2)$. Find the gradient of f at $\mathbf{a}$. Then find the tangent hyperplane to f at this point.

(9) Earlier in the exam, we considered the following equation:

$$x^2 + y^2 + \frac{z^2}{4} = 1.$$

Consider the point on the surface at $(1/2, 0, \sqrt{3})$. Give the *unit* vector for which the directional derivative is maximized at this point and the value of this maximum. Give the *unit* vector for which it is minimized and the value of this minimum. Lastly, give a vector which maintains the z-value as $\sqrt{3}$.

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2 + y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$