MULTIVARIABLE CALCULUS
EXAM 1
SPRING 2024

Total : 60 points

Avg : 51 points.

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: You may OMIT one **5-point** question; indicate which you omit by putting a slash through it. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. A formula sheet is on the last page. Best of luck!

(1) [5 points] Where does the line $x = 1 - 4t, y = t - 3/2, z = 2t + 1$ intersect the plane $5x - 2y + z = 1$?

We substitute into the equation of the plane:

$$5(1-4t) - 2(t - 3/2) + (2t+1) = 1$$

This allows us to solve for t:

$$5 - 20t - 2t + 3 + 2t + 1 = 1$$

$$-20t + 9 = 1$$

$$t = \frac{-8}{-20} = \frac{2}{5}$$

Plugging t into the line equations gives

$$x = 1 - 4\left(\tfrac{2}{5}\right) = \frac{-3}{5}$$

$$y = \frac{2}{5} - \frac{3}{2} = \frac{4}{10} - \frac{15}{10} = \frac{-11}{10}$$

$$z = 2\left(\tfrac{2}{5}\right) + 1 = \frac{9}{5}$$

Date: π day, 2024.

Thus, the line intersects the plane at

$$(-3/5, -11/10, 9/5).$$

(2) [5 points] Show that the vectors $||\mathbf{b}||\mathbf{a}+||\mathbf{a}||\mathbf{b}$ and $||\mathbf{b}||\mathbf{a}-||\mathbf{a}||\mathbf{b}$ are orthogonal where $\mathbf{a}=(3,4,12)$ and $\mathbf{b}=(1,0,1)$.

Two vectors are orthogonal if their dot product is zero. So, let's show this true.

First we compute

$$||\vec{a}|| = \sqrt{3^2+4^2+12^2} = \sqrt{169} = 13$$

$$||\vec{b}|| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

So, the first vector is:

$\sqrt{2}\,\vec{a} + 13\vec{b}$ and the second is $\sqrt{2}\,\vec{a} - 13\vec{b}$.

The dot product is:

$$(\sqrt{2}\,\vec{a} + 13\vec{b}) \cdot (\sqrt{2}\,\vec{a} - 13\vec{b}) =$$

$$2\vec{a}\cdot\vec{a} - \sqrt{2}(13)\vec{a}\cdot\vec{b} + \sqrt{2}\,13\,\vec{a}\cdot\vec{b} - 169\,\vec{b}\cdot\vec{b}$$

$$= 2||\vec{a}||^2 - 169||\vec{b}||^2$$

$$= 2\cdot 13^2 - 169(\sqrt{2})^2 = 0.$$

(4) [5 points] Find the values for A so that the planes

$$Ax - y + z = 1$$

and

$$3Ax + Ay - 2z = 5$$

are perpendicular.

The vector that is normal to the first plane is $(A, -1, 1)$ and the vector that is normal to the second plane is $(3A, A, -2)$.

For the planes to be perpendicular, their normal vectors must be perpendicular, which occurs if the dot products of these is 0.

So,

$$(A, -1, 1) \cdot (3A, A, -2) = 3A^2 - A - 2$$

This factors as $(3A + 2)(A - 1)$.

This equals zero when $3A + 2 = 0$ or $A - 1 = 0$

$$A = -\frac{2}{3} \qquad\qquad A = 1$$

(3) [10 points] The volume of a parallelepiped is given by two formulas:

$$||\mathbf{a} \times \mathbf{b}||||\mathbf{c}|||\cos(\theta)|$$

and

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.$$

(a) Compute the volume of the parallelepiped when $\mathbf{a} = (1,0,1), \mathbf{b} = (1,1,1)$ and $\mathbf{c} = (0,0,1)$.

We prefer the ease of computation of the 2nd formula.

We first compute $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} - 0\vec{j} + \vec{k} \Rightarrow (-1,0,1)$

$|(-1,0,1) \cdot (0,0,1)| = 1$. Thus, the volume is 1.

(b) What is θ referring to in the formula?

The angle between the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$

(c) What are the set of possible values of θ?

$0 \leq \theta \leq \pi$

(d) For which values of θ can the absolute value sign be "dropped"?

For the values where $\cos\theta$ is positive or zero, i.e. $0 \leq \theta \leq \pi/2$

(e) If the absolute valued sign was "dropped" for a value of θ for which it shouldn't, what non-sensical result would be obtained?

A negative volume.

(6) [10 points] Consider the following equation:

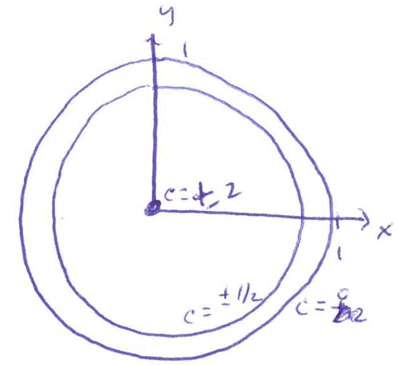
$$x^2 + y^2 + \frac{z^2}{4} = 1.$$

(a) Sketch several level curves of this surface.

Solving for z^2: $z^2 = 4 - 4x^2 - 4y^2$

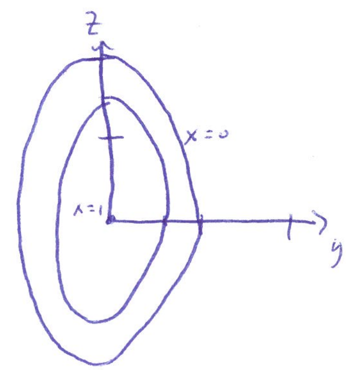
circles $\left\{ \begin{array}{ll} \text{If } z = 0: & 0 = 4 - 4x^2 - 4y^2 \Rightarrow x^2 + y^2 = 1 \\ z = \pm\frac{1}{2}: & x^2 + y^2 = \frac{15}{16} \\ z = \pm 2: & x^2 + y^2 = 0 \end{array} \right.$

$|z| > 2$: nothing.



(b) Sketch sections of the surface by planes of the form $x = c$.

ellipses. $\left\{ \begin{array}{ll} \text{If } x = 0: & y^2 + \frac{z^2}{4} = 1 \\ x = 1/2 & y^2 + \frac{z^2}{4} = 3/4 \Rightarrow \frac{y^2}{3/4} + \frac{z^2}{3} = 1 \\ x = 1 & y^2 + \frac{z^2}{4} = 0 \end{array} \right.$



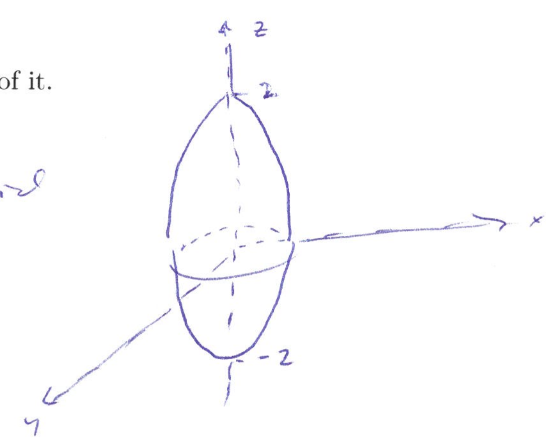
(c) Sketch sections of the surface by planes of the form $y = c$.

Symmetric to the previous

symmetric to previous.

(d) Describe the surface or make a sketch of it.

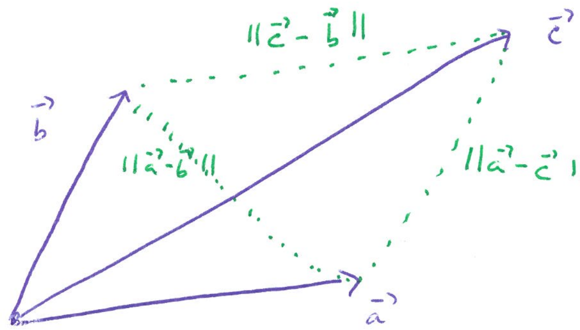
This is an ellipsoid



(5) [5 points] It is true that for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, we have

$$||\mathbf{a} - \mathbf{b}|| \leq ||\mathbf{a} - \mathbf{c}|| + ||\mathbf{c} - \mathbf{b}||.$$

Illustrate this inequality. That is, draw a picture that highlights the important aspects of the inequality. If you were to name this inequality, what name would you give it?



How about "The Triangle Inequality"!

To go beyond what was asked:

$$||\vec{a} - \vec{b}|| = ||(a-c) + (c-b)|| \leq ||\vec{a} - \vec{c}|| + ||\vec{c} - \vec{b}||$$

(8) [10 points]. Given $f(x, y, z) = \sin(xyz)$ and the point $\mathbf{a} = (\pi, 0, \pi/2)$. Find the gradient of f at $\mathbf{a}$. Then find the tangent hyperplane to f at this point.

$$\nabla f = \left(yz\cos(xyz),\ xz\cos(xyz),\ xy\cos(xyz) \right)$$

$$\nabla f(\vec{a}) = \left(0,\ \frac{\pi^2}{2}\cos(0),\ 0 \right)$$

$$= \left(0,\ \frac{\pi^2}{2},\ 0 \right)$$

$$h(\vec{x}) = f(\vec{a}) + f_x(\vec{a})(x - a_1) + f_y(\vec{a})(y - a_2) + f_z(\vec{a})(z - a_3)$$

$$h(\vec{x}) = \sin(0) + 0 + \frac{\pi^2}{2}(y - 0) + 0$$

$h(\vec{x}) = \frac{\pi^2}{2} y$ is the tangent hyperplane.

(7) [5 points] Show that the following limit does not exist by computing the limits along the lines $y = x$ and another line of your choosing.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

If $y = x$, then the limit is

$$\lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

If $y = 2x$, then the limit is

$$\lim_{x\to 0} \frac{x^2}{5x^2} = \frac{1}{5}$$

The limits are not equal, thus the limit does not exist.

(9) Earlier in the exam, we considered the following equation:

$$x^2 + y^2 + \frac{z^2}{4} = 1.$$

Consider the point on the surface at $(1/2, 0, \sqrt{3})$. Give the *unit* vector for which the directional derivative is maximized at this point and the value of this maximum. Give the *unit* vector for which it is minimized and the value of this minimum. Lastly, give a vector which maintains the z-value as $\sqrt{3}$.

According to Theorem 6.3, the directional derivative is maximized when $\vec{u}$ points in the same direction as $\nabla f(\vec{a})$.

Note that the point $(1/2, 0, \sqrt{3})$ "sits" on the "upper portion" of the ellipsoid. Thus, we can consider

$$z = +\sqrt{4 - 4x^2 - 4y^2} = f(x,y)$$

$$\nabla f(x,y) = \left(\tfrac{1}{2}(4 - 4x^2 - 4y^2)^{-1/2} \cdot (-8x),\ \tfrac{1}{2}(4 - 4x^2 - 4y^2)^{-1/2}(-8y) \right)$$

$$\nabla f(\tfrac{1}{2}, 0) = \left(\tfrac{1}{2}\left(4 - 4 \cdot \tfrac{1}{4}\right)^{-1/2}(-4),\ 0 \right)$$

$$= \left(-\frac{2}{\sqrt{3}},\ 0 \right)$$

We normalize this vector $\vec{u} = \dfrac{1}{\|(-\frac{2}{\sqrt{3}}, 0)\|}\left(-\frac{2}{\sqrt{3}}, 0\right) = (-1, 0)$

$(-\frac{2}{\sqrt{3}}, 0) \cdot (-1, 0) = \frac{2}{\sqrt{3}}$

Thus $(-1,0)$ maximizes the directional derivative w/ max value ~~(-1,0)~~

Thus, $(+1, 0)$ minimizes the directional derivative w/ min value $-\frac{2}{\sqrt{3}}$

And, $(0,1)$ and $(0,-1)$ maintains the z-value as $\sqrt{3}$.

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ \ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$