# MULTIVARIABLE CALCULUS
# EXAM 1
# SPRING 2021

**Name:** Solution Key

**Honor Code Statement:** I have neither given nor received unauthorized aid.

**Directions:** You may OMIT one five point question; indicate which you omit by putting a slash through it. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. A formula sheet is on the last page. Best of luck!

60 points total

Average 51.5

(1) [5 points] Write a set of parametric equations for the line in $\mathbb{R}^4$ through the points $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$.

The line is in the direction of the vector $(1,2,3,4) - (1,1,1,1) = (0,1,2,3)$ and contains $(1,1,1,1)$.

Thus, $\vec{r}(t) = (1,1,1,1) + t(0,1,2,3)$.

Thus
$$x_1 = 1$$
$$x_2 = 1 + 1t$$
$$x_3 = 1 + 2t$$
$$x_4 = 1 + 3t$$

*Date*: March 25, 2021.

(2) [5 points] Find three vectors that are perpendicular to $\mathbf{i} - \mathbf{j} + \mathbf{k}$ such that no two of these are parallel.

The vectors that are perpendicular to the given one all "sit" in the same plane for which the given is the normal vector. Its equation is

$$1x - 1y + 1z = 0$$

From this we see that $(1,1,0)$, $(1,0,-1)$ and $(0,1,1)$ are three such vectors, no one of which is a scalar multiple of another.

(3) [5 points] Find the volume of the parallelepiped determined by

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j}, \ \mathbf{b} = -2\mathbf{i} + \mathbf{k}, \ \mathbf{c} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

The volume of a parallelepiped is given by

$$|(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

So, $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -1\vec{i} - 3\vec{j} \cancel{+} - 2\vec{k}$$

Thus, $$|(-\vec{i} - 3\vec{j} \cancel{+} - 2\vec{k}) \cdot (\vec{i} - 2\vec{j} + 4\vec{k})| = |-1 + 6 \cancel{+} - 8| = \cancel{13}\ 3$$

(4) [5 points] Calculate the distance between the two planes:

$$3x - 3y + 2z = 8, \;\; -12x + 12y - 8z = 0.$$

Note these planes are parallel.

A point $P_1$ on the first plane is $(0,0,4)$

A point $P_2$ on the second plane is $(1,1,0)$.

$\overrightarrow{P_1P_2} = (1,1,-4)$

A vector normal to the planes is $\vec{n} = (3,-3,2)$.

Thus, $\text{proj}_{\vec{n}} \overrightarrow{P_1P_2} = \left(\dfrac{\vec{n}\cdot\overrightarrow{P_1P_2}}{\vec{n}\cdot\vec{n}}\right)\vec{n} = \dfrac{-8}{22}(3,-3,2)$

Thus, the distance is $\left\| \frac{-8}{22}(3,-3,2) \right\|$

$$= \frac{8}{22}\sqrt{9+9+4} = \frac{8\sqrt{22}}{22}$$

(5) [5 points] Let's try a short proof. Let $\mathbf{a}, \mathbf{b}$ be vectors in $\mathbb{R}^n$. Let's assume that $||\mathbf{a}+\mathbf{b}|| = ||\mathbf{a}-\mathbf{b}||$. You will prove that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal by squaring both sides of the given equation, "multiplying out", and collecting like terms. Do this.

$$||\vec{a} + \vec{b}||^2 = ||\vec{a} - \vec{b}||^2$$

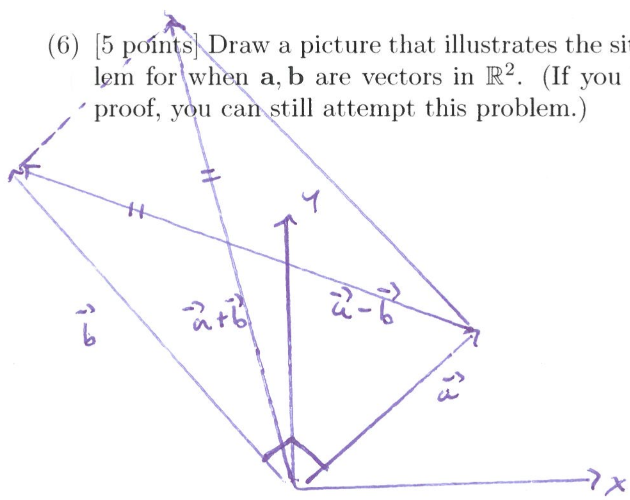
$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$\vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$4\vec{a} \cdot \vec{b} = 0$$

Thus $\vec{a}$ and $\vec{b}$ are orthogonal.

(6) [5 points] Draw a picture that illustrates the situation in the previous problem for when $\mathbf{a}, \mathbf{b}$ are vectors in $\mathbb{R}^2$. (If you haven't yet given the above proof, you can still attempt this problem.)




We have $\vec{a}$ and $\vec{b}$ orthogonal. The parallelogram they form has diagonals equal in length.

(8) [5 points] Show that the following limit does not exist.

$$\lim_{(x,y)\to(2,0)} \frac{x^2 - y^2 - 4x + 4}{x^2 + y^2 - 4x + 4}.$$

Let's consider the limit along the line $x = 2$

Then we have

$$\lim_{y \to 0} \frac{4 - y^2 - 8 + 4}{4 + y^2 - 8 + 4} = \lim_{y \to 0} \frac{-y^2}{y^2} = -1$$

Let's consider the limit along the line $y = 0$.

Then we have

$$\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 4x + 4} = 1.$$

Since these values differ, the limit does not exist.

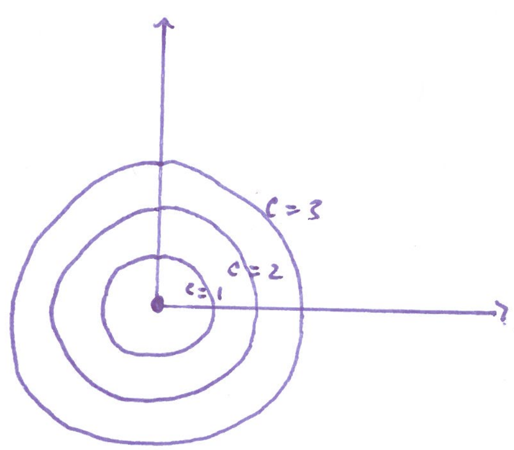
(7) [10 points] Given the function $f(x,y) = \sqrt{x^2+y^2}$ sketch 3 level curves, (sketch two sections of the graph of $f$ by the plane $x = c$, and sketch two sections of the graph of $f$ by the plane $y = c$.) Sketch **or** describe the surface.

Note there are no level curves for $c < 0$ since $x^2+y^2$ is always non-negative.

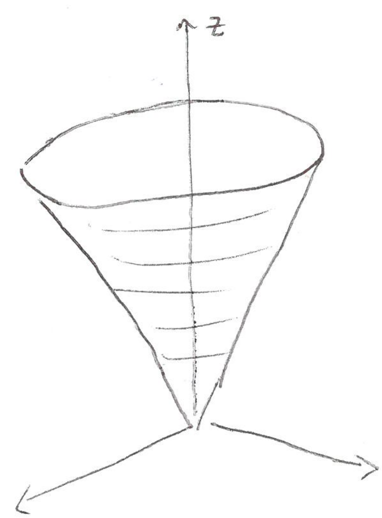
So, if

| $c$ | $f$ |
|---|---|
| 0 | $0 = \sqrt{x^2+y^2} \Rightarrow 0 = x^2+y^2$ |
| 1 | $1 = \sqrt{x^2+y^2} \Rightarrow 1 = x^2+y^2$ |
| 2 | $2 = \sqrt{x^2+y^2} \Rightarrow 4 = x^2+y^2$ |
| 3 | $3 = \sqrt{x^2+y^2} \Rightarrow 9 = x^2+y^2$ |

these are circles.




The surface is a cone that is "open" to the "sky" and with its "point" at the origin.



(9) [10 points] Give the equation of the tangent plane of the given function at the given point.

$$f(x,y) = ((x-1)y)^{2/3},\ (a,b) = (1,0)$$

First, we compute the gradient of $f$

$$\nabla f(x,y) = \left( \frac{2}{3} y \left((x-1)y\right)^{-1/3},\ \frac{2}{3}(x-1)\left((x-1)y\right)^{-1/3} \right)$$

Note that these partial derivatives are not continuous (or defined) at $(1,0)$. However, this does not mean that $f$ is not differentiable. (Carefully re-read Theorem 3.5!)

Note that the partial functions at $(1,0)$ are

$$f(x,0) = 0 \quad \text{and} \quad f(1,y) = 0$$

Thus, $f_x(1,0) = 0$ and $f_y(1,0) = 0$.

So, since $f(1,0) = 0$ the tangent plane's equation is

$$z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

or

$$z = 0 + 0(x-1) + 0(y) = 0$$

Tricky!!

(10) [10 points] Consider the following function at the following point.

$$f(x,y) = e^y \sin x, \mathbf{a} = (\frac{\pi}{3}, 0)$$

(a) Compute the direction of steepest descent.

This is given by the negative of the gradient at the point.

$$\nabla f(x,y) = (e^y \cos x, e^y \sin x)$$

$$\nabla f(\tfrac{\pi}{3}, 0) = (1 \cos\tfrac{\pi}{3}, 1 \sin\tfrac{\pi}{3}) = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})$$

So steepest descent is $(-1/2, -\frac{\sqrt{3}}{2})$

(b) Compute a direction for maintaining the current "altitude".

Need a vector orthogonal to gradient:

$$-\frac{1}{2} x + -\frac{\sqrt{3}}{2} y = 0$$

Let $x = 1$, then $-\frac{\sqrt{3}}{2} y = \frac{1}{2} \Rightarrow y = -\frac{1}{\sqrt{3}}$

Thus, $(x,y) = (1, -\frac{1}{\sqrt{3}})$

(c) Compute the directional derivative in the direction parallel to the vector $\mathbf{u} = \mathbf{i} - \mathbf{j}$.

This vector needs to be a unit vector, so normalize it: $\vec{u} = \frac{i}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

Then $D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$

Then

$$D_{\vec{u}} f(\tfrac{\pi}{3}, 0) = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \cdot (\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}})$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

**Change of coordinates**

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$