MULTIVARIABLE CALCULUS
EXAM 1
SPRING 2018

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. Best of luck.

(1) [10 points] Give an equation for the plane parallel to the plane $2x-3y+z=5$ that passes through the point $(-1,1,2)$. **Also,** find the distance between the given plane and point.

The normal vector to the given plane can be found by "reading" the standard equation; it is $\vec{n} = (2,-3,1)$.

The distance between the point and the plane is the same as the distance between the given plane and the parallel plane containing the given point. This distance is given by $\|\text{proj}_{\vec{n}} \overrightarrow{P_0P_1}\|$, where P_0 is a point in one plane and P_1 a point in the other. $(0,0,5) = P_0$ is a point in the given plane and we know $P_1 = (-1,1,2)$ is a point in the other.

Then $\overrightarrow{P_0P_1} = (-1,1,2) - (0,0,5) = (-1,1,-3)$, and so

$$\text{proj}_{\vec{n}} \overrightarrow{P_0P_1} = \frac{\vec{n} \cdot \overrightarrow{P_0P_1}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{(2,-3,1)\cdot(-1,1,-3)}{(2,-3,1)\cdot(2,-3,1)} (2,-3,1)$$

$$= -\frac{8}{14}(2,-3,1)$$

The norm of this vector is $\left|\frac{-8}{14}\right| \sqrt{2^2+(-3)^2+1^2}$

$$= \frac{8}{14}\sqrt{14}$$

Date: March 15, 2018.

(2) [10 points] **Graph paper is available, which should improve your accuracy and conclusions you make.** Determine and draw several – say, at least four – level curves of the given function f (and make sure to indicate the height c of each curve).

$$f(x, y) = \sqrt{x^2 + y^2}$$

The range of this function is the positive reals.

Let c be a positive real, the $\sqrt{x^2+y^2} = c \Leftrightarrow x^2+y^2=c^2$, which is the equation of a circle of radius c.

We draw 4 level curves (on the graph paper) – these are for $c=0$, $c=1$, $c=2$, $c=3$

Now give and draw four sections of the graph of f by planes of the form $y = c$, where c is a constant.

A section by $y=c$ is $\{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y), y=c\}$.

We make selections for c: $0, 1, 2, 3$, and get

$c=0$ $\quad \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+0} = \sqrt{x^2}\}$

$c=1$ $\quad \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+1}\}$

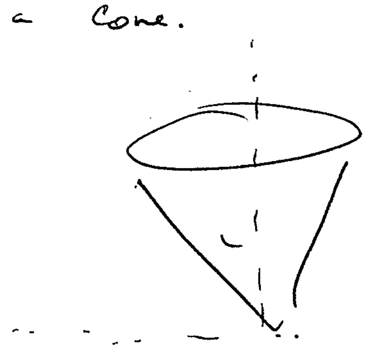
$c=2$ $\quad \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+4}\}$

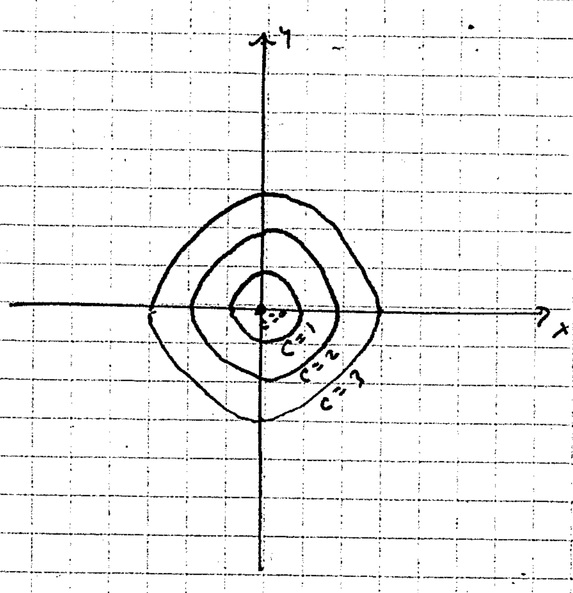
$c=3$ $\quad \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+9}\}$

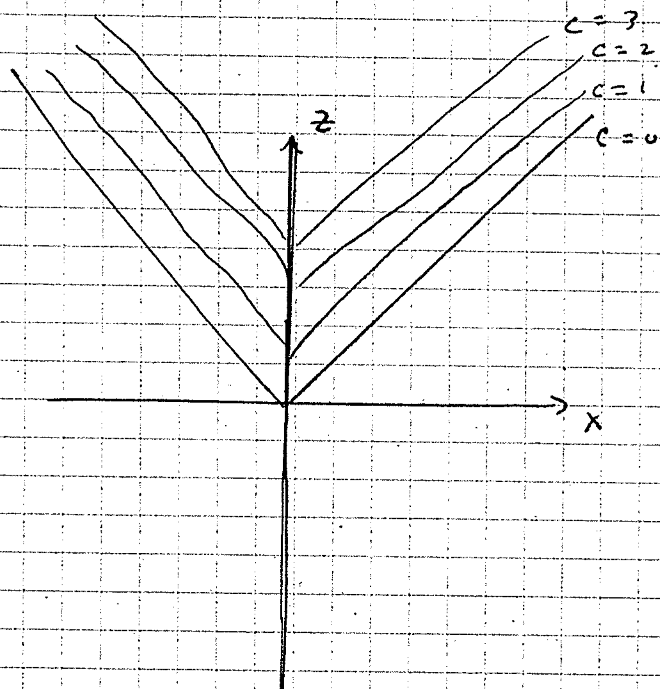
Use the above information to sketch the graph of f or describe it accurately in words.

Imagine yourself at $(0,0)$ on the graph. Moving away from the origin in any direction, we "head upward" at a constant rate. The surface is a cone.

An alternate approach: make a change of coordinate $r^2 = x^2+y^2$ so that $f = \sqrt{r^2} = r$. This helps, yes!




y
x
c=1
c=2
c=3


c = 3
c = 2
c = 1
c = 0
z
x

(3) [5 points] Determine the following limit, if it exists. Hint: make a change of variable.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$$

We follow the hint (and use the change of coordinates formulas on attached sheet): $x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$. Thus the limit becomes

$$\lim_{r\to 0} \frac{r^2+r\cos\theta\, r\sin\theta}{r^2} = \lim_{r\to 0} 1+\cos\theta\sin\theta.$$

This limit depends upon the choice of θ. Thus we do not have uniqueness of limits – the limit does not exist.

(4) [5 points] State what it means for a function to be **continuous**.

As found on page 109 of the text (Definition 2.7)

Let $\vec{f}: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{a} \in X$. Then $\vec{f}$ is said to be <u>continuous at $\vec{a}$</u> if either $\vec{a}$ is an isolated point of X or if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

If $\vec{f}$ is continuous at all points of its domain X, then we say $\vec{f}$ is <u>continuous</u>.

(5) [3 points] Choose the correct term to describe the following set. $S := \{(x, y, z) \in \mathbb{R}^3 | 1 < x^2 + y^2 < 4\}$. Terms: open, closed or neither. Justify your answer.

S is open since for any point in S there is an open ball centered at that point entirely within S.

~~(6) [3 points] True or false? There exists two vectors in S whose cross product is also in S. Justify your answer.~~

(7) [4 points] Do there exist $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}^2$ with $||p - q|| = ||q - r|| = 1$ and $||p - r|| = 3$?

Consider this representative drawing:

p and q are at distance 1, q and r are at distance 1, and so the Triangle Inequality says that the distance from p to r cannot be more than 2.

(8) [10 points] Consider the graph $z = x^3 - 7xy + e^y$. Find the tangent plane at $(-1, 0, 0)$. Is this tangent plane a good linear approximation to the graph near the point of tangency? Why?

Let $f(x,y) = x^3 - 7xy + e^y$, then the tangent plane, if it exists, has equation

$$z = f(-1,0) + f_x(-1,0)(x+1) + f_y(-1,0)(y-0)$$

So, we compute the first-partial derivatives:

$$f_x = 3x^2 - 7y$$
$$f_y = -7x + e^y$$

Note that these partial derivatives are continuous everywhere, and it is this that will guarantee differentiability (by Theorem 3.5), i.e. that the tangent plane is a good linear approximation to the graph.

$$f_x(-1,0) = 3(-1)^2 - 7\cdot 0 = 3$$
$$f_y(-1,0) = -7(-1) + e^0 = 8.$$

We are told $f(-1,0) = 0$.

Thus, the tangent plane has equation

$$z = 0 + 3(x+1) + 8(y-0) = 3(x+1) + 8y$$

(9) [5 points] *Use the product rule* to find the derivative of the product of the following two functions: $f(x, y, z) = e^{x+y}$ and $g(x, y, z) = xy + xz + yz$.

$$Df(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} e^{x+y} & e^{x+y} & 0 \end{bmatrix}$$

$$Dg(x,y,z) = \begin{bmatrix} g_x & g_y & g_z \end{bmatrix} = \begin{bmatrix} y+z & x+z & x+y \end{bmatrix}$$

Thus, $D(fg)(x,y,z) = g(x,y,z)Df(x,y,z) + f(x,y,z)Dg(x,y,z)$

$$= (xy+xz+yz)\begin{bmatrix} e^{x+y} & e^{x+y} & 0 \end{bmatrix} + e^{x+y}\begin{bmatrix} y+z & x+z & x+y \end{bmatrix}$$

$$= \begin{bmatrix} (xy+xz+yz)e^{x+y} & (xy+xz+yz)e^{x+y} & 0 \end{bmatrix} + \begin{bmatrix} e^{x+y}(y+z) & e^{x+y}(x+z) & e^{x+y}(x+y) \end{bmatrix}.$$

$$= \begin{bmatrix} e^{x+y}(xy+xz+yz+y+z) & e^{x+y}(xy+xz+yz+x+z) & e^{x+y}(x+y) \end{bmatrix}.$$

(10) [10 points] Suppose that a bird flies along the helical curve $x = 2\cos t, y = 2\sin t, z = 3t$. The bird suddenly encounters a weather front so that the barometric pressure is varying rather wildly from point to point as

$$P(x,y,z) = \frac{6x^2z}{y}atm.$$

Use the chain rule to determine how the pressure is changing at $t = \frac{\pi}{4}$.

We have $\vec{x}(t) = (2\cos t, 2\sin t, 3t)$

By the Chain Rule, we have

$$\frac{dP}{dt}(t_0) = \nabla P(\vec{x}(t_0)) \cdot \vec{x}\,'(t_0)$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt}$$

$$= \frac{12xz}{y}(-2\sin t) + \left(\frac{-6x^2z}{y^2}\right)(2\cos t) + \left(\frac{6x^2}{y}\right)\cdot 3$$

$$= \frac{12(2\cos t)(3t)}{2\sin t}(-2\sin t) + \frac{(-6)\,4\cos^2 t\cdot 3t}{4\sin^2 t}\cdot 2\cos t + \frac{6\cdot 4\cos^2 t}{2\sin t}\cdot 3$$

$$= -72\,t\cos t - \frac{36t\cos^3 t}{\sin^2 t} + \frac{36\cos^2 t}{\sin t}$$

We evaluate at $t = \pi/4$ to get

$$-72\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} - \frac{36\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right)^3}{\left(\frac{\sqrt{2}}{2}\right)^2} + \frac{36\left(\frac{\sqrt{2}}{2}\right)^2}{\sqrt{2}/2}$$

$$= -9\pi\sqrt{2} - \frac{18\sqrt{2}.\pi}{4} + 18\sqrt{2} = \frac{-27\pi\sqrt{2}}{2} + 18\sqrt{2}$$

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$