MULTIVARIABLE CALCULUS
EXAM 1
FALL 2019

**Name:** Answer Key
**Honor Code Statement:** I have neither given nor received unauthorized aid on this exam.

**Directions:** Complete all problems. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. A formula sheet is on the last page. Best of luck.

(1) [5 points] Find three nonparallel vectors that are perpendicular to $\mathbf{i}-\mathbf{j}+\mathbf{k}$.

Any vector in the plane that has $\vec{i}-\vec{j}+\vec{k}$ as the normal vector and passes thru the origin $(0,0,0)$ will be a suitable vector. Then we'll find 3 of these that are not parallel.

The equation for the plane is: $1x - 1y + 1z = 0$

Then $(1,1,0)$, $(0,1,1)$ and $(-1,0,1)$ will work.

(2) [5 points] Use the cross product to find the area of the triangle determined by the vectors $\mathbf{a} = (1,-2,6)$ and $\mathbf{b} = (4,3,-1)$.

The norm of the cross product of these 2 vectors gives the area of their parallelogram. The triangle will be half of this.

We compute the cross product as a determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 6 \\ 4 & 3 & -1 \end{vmatrix} = \vec{i}(-16) - \vec{j}(-25) + \vec{k}(11)$$

The norm of this vector is

$$\sqrt{(-16)^2+(25)^2+(11)^2}$$

The area of the triangle is $\frac{1}{2}\sqrt{16^2+25^2+11^2}$

$\sqrt{1002}$

*Date*: October 10, 2019.

(3) [5 points] Calculate the distance between the two parallel planes

$$5x - 2y + 2z = 12$$

and

$$-10x + 4y - 4z = 8$$

.

The point $P_1 = (0, 0, 6)$ is on the first plane, and $P_2 = (0, 0, -2)$ is on the second. $\overrightarrow{P_1P_2} = (0, 0, -8)$. A normal vector to plane 1 is $(5, -2, 2)$. So the distance between the two planes is $\|\text{proj}_{\vec{n}} \overrightarrow{P_1P_2}\| = \left\| \frac{(0,0,-8)\cdot(5,-2,2)}{(5,-2,2)\cdot(5,-2,2)} (5,-2,2) \right\|$

$$= \left\| \frac{-16}{33} (5, -2, 2) \right\|$$

(4) [2 points] In one sentence, explain how one can tell these two given equations yield *parallel* planes.

We can "read" the normal vectors to these planes from the given equations as $(5, -2, 2)$ and $(-10, 4, -4)$, which are parallel and thus the planes are parallel.

(5) [6 points] The following expressions are "nonsense", that is, these are undefined. Make the smallest number of changes possible to each expression, restricting yourself to the insertion, deletion or replacement of a binary operation, so that the expression now has meaning. Indicate whether your new expression is a scalar, a vector or something else.

I give Some possibilities

- $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$

$(\vec{a} \times \vec{b}) \cdot \vec{c}$ is a scalar

$(\vec{a} + \vec{b}) \cdot \vec{c}$ is a scalar

- $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$

$(\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d})$ is a scalar

$(\vec{a} \cdot \vec{b}) + (\vec{c} \cdot \vec{d})$ is a scalar

- $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is ~~a scalar~~ a vector

(6) [4 points] Write what it means to say for a scalar-valued function $f$ to be **differentiable** at $\mathbf{a} = (a_1, \ldots, a_n)$.

For $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\vec{a} \in X$, $f$ is differentiable at $\vec{a}$ if all the partial derivatives exist and if

$$h(\vec{x}) = f(\vec{a}) + f_{x_1}(\vec{a})(x_1 - a_1) + \ldots + f_{x_n}(\vec{a})(x_n - a_n)$$

is a good linear approximation of $f$ near $\vec{a}$.

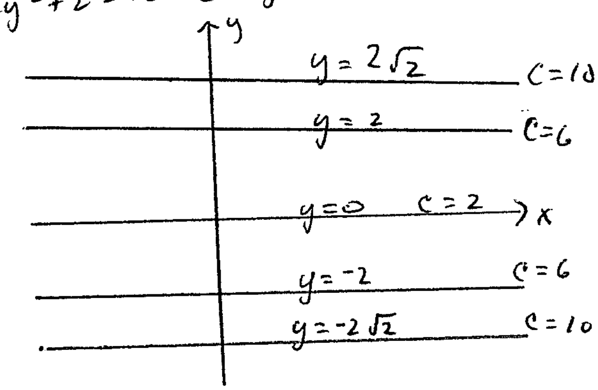
(7) [10 points] Consider the surface in $\mathbb{R}^3$ determined by the following equation, $z = y^2 + 2$.. (Yes, you read it correctly. This is a surface in $\mathbb{R}^3$ and $x$ does not appear in the equation.) Sketch 3 level curves. Sketch two sections of the graph of $f$ by the plane $x = c$. Sketch two sections of the graph of $f$ by the plane $y = c$. Sketch **or** describe the surface.

As $y^2+2 \geq 2$, it only makes sense to consider level curves of height at least 2.

At $c=2$, $y^2+2=2 \Leftrightarrow y^2=0 \Leftrightarrow y=0$

at $c=6$, $y^2+2=6 \Leftrightarrow y^2=4 \Leftrightarrow y=\pm 2$.

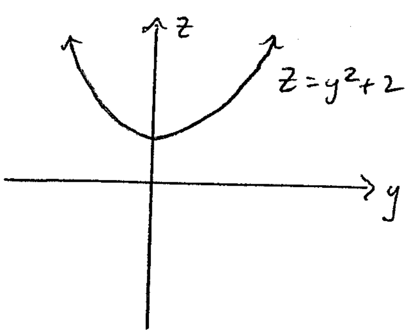
$c=10$ $y^2+2=10 \Leftrightarrow y^2=8 \Leftrightarrow y=\pm 2\sqrt{2}$




For sections with planes $x=c$:

regardless of the choice of $c$, we get

$z=y^2+2$

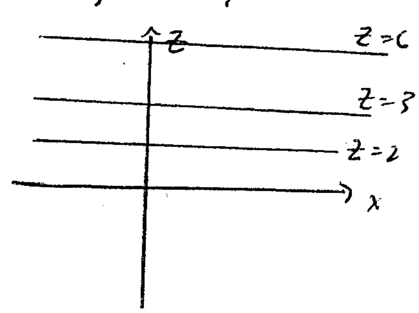



For sections with planes $y=c$

if $y=0$, $z=2$

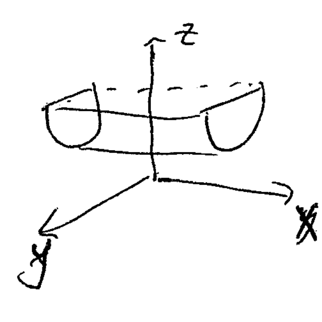
if $y=\pm 1$, $z=3$

if $y=\pm 2$, $z=6$




Putting it together

The surface looks like a parabolic "trough" at altitude 2.



(8) [6 points] Calculate the following limit by first ~~change~~ changing from Cartesian coordinates to polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$$

Our change of coordinates is $x = r\cos\theta$, $y = r\sin\theta$

So the limit becomes

$$\lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$ since $r \to 0$ as $(x,y)\to(0,0)$

This limit simplifies:

$$\lim_{r\to 0^+} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r} = \lim_{r\to 0^+} r\left(\cos^2\theta - \sin^2\theta\right)$$

$$= \lim_{r\to 0^+} r\, 2\cos\theta = 0$$

since $-r \le r\cos 2\theta \le r$

(9) [6 points] Find an equation for the hyperplane tangent to the 4-dimensional paraboloid

$$x_5 = 10 - (x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2)$$

at the point $(2, -1, 1, 3, 8)$. my typo! should be $-8$

The equation of the tangent hyperplane is

$$\vec{x_5} = f(\vec{a}) + f_{x_1}(\vec{a})(x_1 - a_1) + f_{x_2}(\vec{a})(x_2 - a_2) + f_{x_3}(\vec{a})(x_3 - a_3) + f_{x_4}(\vec{a})(x_4 - a_4)$$

where $\vec{a} = (2, -1, 1, 3)$ and $f(\vec{x}) = 10 - (x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2)$

We find $\nabla f(\vec{x}) = (-2x_1, -6x_2, -4x_3, -2x_4)$

$$\nabla f(\vec{a}) = (-4, 6, -4, -6)$$

$$f(\vec{a}) = -8 \quad \text{(as given)}$$

So, we find

$$\vec{x_5} = -8 + -4(x_1 - 2) + 6(x_2 + 1) - 4(x_3 - 1) - 6(x_4 - 3)$$

$$= -8 - 4x_1 + 8 + 6x_2 + 6 - 4x_3 + 4 - 6x_4 + 18$$

$$= -4x_1 + 6x_2 - 4x_3 - 6x_4 + 28$$

(10) [6 points] Fill in the blank. Suppose that two surfaces are given by the equations $F(x, y, z) = c$ and $G(x, y, z) = k$. Further, suppose that the surfaces intersect at the point $(x_0, y_0, z_0)$. The surfaces are tangent at $(x_0, y_0, z_0)$ if and only if the tangent planes at $(x_0, y_0, z_0)$ are the ___Same___ if and only if the ___normal___ vectors at $(x_0, y_0, z_0)$ are parallel if and only if $\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$ equals ___the zero vector___

(11) [5 points] Find the directional derivative of $f(x, y, z) = \frac{xe^y}{3z^2+1}$ at the point $\mathbf{a} = (2, -1, 0)$ in the direction parallel to the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Is there a direction for which the directional derivative is larger?

Need to normalize this!

By Theorem 6.2 of Section 2.6, we can compute the directional derivative as

$$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$ where $\vec{u}$ is a unit vector

So, $$\nabla f = \left( \frac{e^y}{3z^2+1}, \frac{xe^y}{3z^2+1}, \frac{-6zxe^y}{(3z^2+1)^2} \right)$$

and $$\nabla f(\vec{a}) = \left( \frac{e^{-1}}{1}, \frac{2e^{-1}}{1}, 0 \right) = \left( \frac{1}{e}, \frac{2}{e}, 0 \right).$$

So $$D_{\vec{u}} f(\vec{a}) = \left( \frac{1}{e}, \frac{2}{e}, 0 \right) \cdot \frac{(1, -2, 3)}{\sqrt{14}} = \frac{1}{e\sqrt{14}} - \frac{4}{e\sqrt{14}} + 0 = \frac{-3}{e\sqrt{14}}$$

Part 2: Yes, the directional derivative is larger in the direction of the gradient.

(12) [5 points] The following page shows a "proof" of the Cauchy-Schwarz Inequality. One of the inequalities is circled. Explain why this circled inequality is true.**3 sentences maximum**

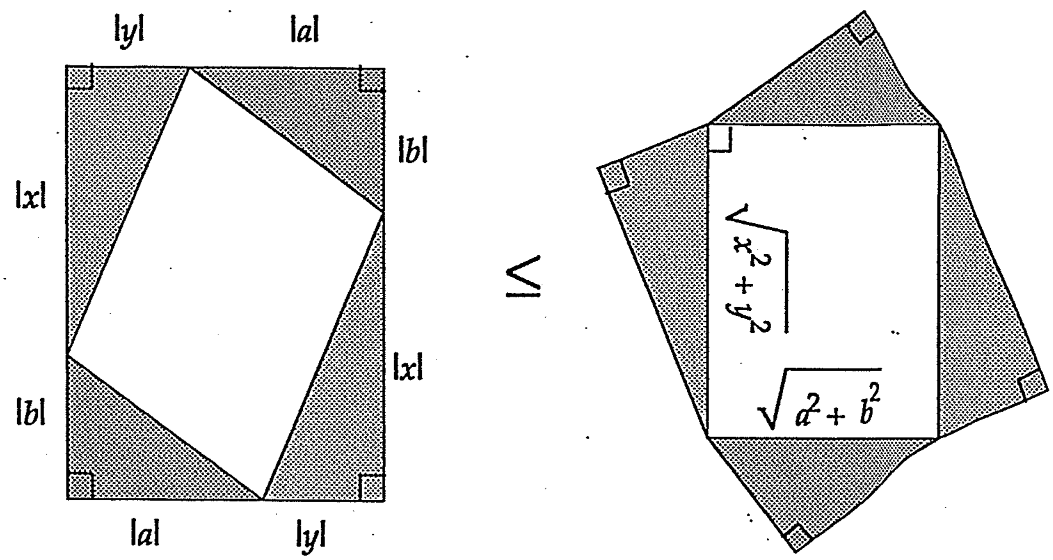
The absolute value of a sum is less than or equal to the sum of the absolute values. Thus,

$$|ax + by| \leq |ax| + |by|.$$

Now $|ax| = |a||x|$ and $|by| = |b||y|$

# The Cauchy-Schwarz Inequality

$$|\langle a,b\rangle \cdot \langle x,y\rangle| \le \|\langle a,b\rangle\| \, \|\langle x,y\rangle\|$$




$$(|a|+|y|)(|b|+|x|) \le 2\left(\tfrac{1}{2}|a||b| + \tfrac{1}{2}|x||y|\right) + \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

$$\therefore\ |ax+by| \le |a||x| + |b||y| \le \sqrt{a^2+b^2}\sqrt{x^2+y^2}.$$

—RBN

## Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$