MULTIVARIABLE CALCULUS
EXAM 1
FALL 2018

Name: Solution Key

60 points total

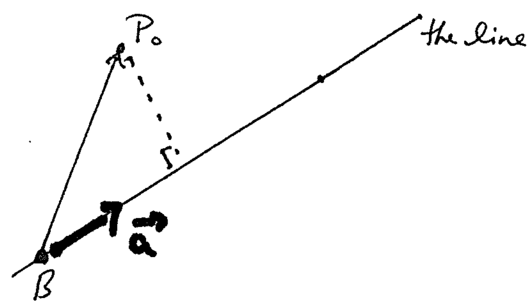
Average: 45 points

Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. A formula sheet is on the last page. Best of luck.

8 points (1) Determine the distance between the point $P_0 = (2, 2, 4)$ and the line $\ell(t) = t(1, 1, 1) + (4, 0, 0)$.

A point on the line is $B = (4, 0, 0)$. A vector parallel to the line is $\vec{a} = (1, 1, 1)$. Consider the scenario as drawn:



We seek the length of the dashed line, which corresponds to the length of the vector $\vec{BP_0} - \text{proj}_{\vec{a}} \vec{BP_0}$.

So, $\vec{BP_0} = (2, 2, 4) - (4, 0, 0) = (-2, 2, 4)$

$$\text{proj}_{\vec{a}} \vec{BP_0} = \left(\frac{\vec{a} \cdot \vec{BP_0}}{\vec{a} \cdot \vec{a}}\right)\vec{a} = \frac{(1,1,1) \cdot (-2,2,4)}{(1,1,1) \cdot (1,1,1)}(1,1,1) = \frac{4}{3}(1,1,1) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

$$\vec{BP_0} - \text{proj}_{\vec{a}} \vec{BP_0} = (-2, 2, 4) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = \left(-\frac{10}{3}, \frac{2}{3}, \frac{8}{3}\right)$$

Date: October 10, 2018.

Its norm is $\sqrt{\frac{100}{9} + \frac{4}{9} + \frac{64}{9}} = \sqrt{\frac{168}{9}} = \frac{2\sqrt{42}}{3}$

10 points (2) Consider the graph of

$$f(x,y) = z = x^2 + y^2 + 2xy$$

and the point $(1,1,4)$.

(a) Give an equation for the tangent plane at the given point.
(b) Is the function differentiable at this point? Why?
(c) What is the direction of steepest ascent at this point on this graph?

(a) We use Theorem 3.3 of Section 2.3, which tells us that at (a,b) the tangent plane has equation

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So we find: $f(1,1) = 4$

$f_x = 2x + 2y \Rightarrow f_x(a,b) = f_x(1,1) = 4$

$f_y = 2y + 2x \Rightarrow f_y(a,b) = f_y(1,1) = 4$

Thus, $z = 4 + 4(x-1) + 4(y-1) = -4 + 4x + 4y$

(b) Theorem 3.10 tells us that f is differentiable at $(1,1)$ if the partial derivatives exist and are continuous in a neighborhood of $(1,1)$. Note that $f_x = 2x + 2y$, $f_y = 2y + 2x$ exist, are polynomials and thus continuous everywhere. We conclude that this f is differentiable at $(1,1)$.

(c) Theorem 6.3 of Section 2.6 tells us that steepest ascent occurs in the direction of the gradient:

$$\nabla f(1,1) = (4, 4).$$

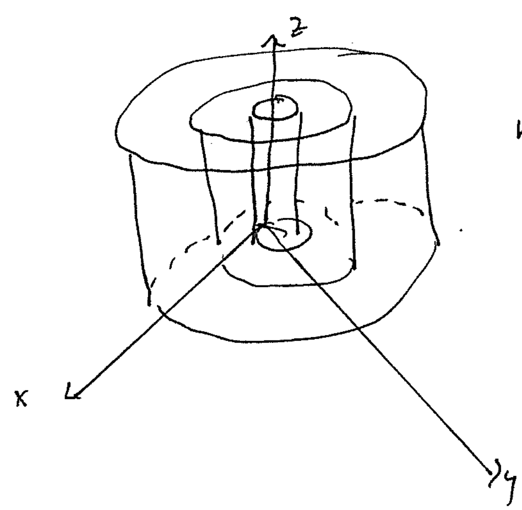
5 points (3) In what vector space does the graph of $g(x, y, z) = x^2 + y^2$ "live"? ____________________

Describe or sketch the level *surfaces* of this graph when $c = 1, 4$ and 9.

The graph of $g(x,y,z)$ is the set of all points (x,y,z,w) s.t. $g(x,y,z) = w$. These points are in $\mathbb{R}^4$.

The level surface at $c=1$ is the set of $(x,y,z) \in \mathbb{R}^3$ s.t. $g(x,y,z) = 1$, i.e. $x^2+y^2=1$. This is a cylinder of radius 1, centered along the z-axis.

For $c=4$, we have a cylinder of radius 2 centered along the z-axis. Similarly, for $c=9$ a cylinder along the z-axis of radius 3.



nested cylinders

* Many people thought that the level surfaces were circles, which is <u>not</u> the case.

10 points (4) (a) State the Cauchy-Schwarz Inequality.
(b) State the Triangle Inequality.
(c) Give a pair of vectors in $\mathbb{R}^2$ that show equality can be achieved in the Cauchy-Schwarz Inequality while simultaneously **not** achieving equality in the Triangle Inequality. Show the calculations that support your claim.

(a) For $\vec{a}, \vec{b} \in \mathbb{R}^n$, we have $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \, \|\vec{b}\|$

(b) For $\vec{a}, \vec{b} \in \mathbb{R}^n$, we have $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

(c) To achieve equality in C.S. Inequality, the vectors must point in the same direction or opposite directions. If the vectors point in the same direction, then they achieve equality in the Triangle Inequality. Thus, we must choose two vectors that point in opposite directions. I'll choose $\vec{a}=(1,0)$ and $\vec{b}=(-1,0)$.

Note $|\vec{a} \cdot \vec{b}| = |(1,0) \cdot (-1,0)| = |1+0| = 1$

$\|\vec{a}\| = 1$, $\|\vec{b}\| = 1$ and $1 \cdot 1 = 1$.

Thus, equality in C.S.

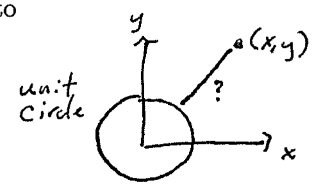
However,

$\vec{a} + \vec{b} = (0,0)$ and $\|\vec{a}+\vec{b}\| = 0$

but $\|\vec{a}\| + \|\vec{b}\| = 2$, and $0 < 2$.

6 points (5) [Problem from O. Knill]

- Give a formula $f(x,y)$ for the distance in the plane from a point (x,y) to the unit circle centered at the origin. (Hint: you may wish to consider two cases.)
- Show that it satisfies the Eiconal equation $f_x^2 + f_y^2 = 1$.



Case (1) For points outside the circle, we have

$$f(x,y) = \|(x,y)\| - 1 = \sqrt{x^2+y^2} - 1$$

Case (2) For points inside or on the circle, we have

$$f(x,y) = 1 - \|(x,y)\| = 1 - \sqrt{x^2+y^2}$$

Case 1

$$f_x = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow f_x^2 + f_y^2 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$

Case 2

Similar to case (1).

6 points (6)
- What is the cross product of the two vectors $\mathbf{a} = (1,1,1)$ and $\mathbf{b} = (2,5,2)$?
- What are the spherical coordinates of $(x, y, z) = (1, \sqrt{3}, 2)$?

(1) We compute the cross product of two vectors using a determinant calculation (avoiding the use of the definition of cross product) as follows:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 5 & 2 \end{vmatrix} = \vec{i}(2-5) - \vec{j}(2-2) + \vec{k}(5-2)$$

$$= -3\vec{i} + 3\vec{k}$$

(2) We use the formulas (gifted to you) on the last page:

$$\rho^2 = 1^2 + (\sqrt{3})^2 + 2^2 = 1+3+4 = 8 \Rightarrow \rho = 2\sqrt{2}$$

$$\tan\varphi = \frac{\sqrt{1+3}}{2} = 1 \Rightarrow \varphi = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \tan^{-1}(\sqrt{3}) \Rightarrow \theta = \frac{\pi}{3}$$

8 points (7) Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$$

(a) Evaluate the limit along the line $x = 0$.
(b) Evaluate the limit along the line $y = 0$.
(c) Evaluate the limit along the parabola $x = y^2$.
(d) What can you conclude about this limit?

(a) We get $\lim_{y\to 0} \frac{0}{(0+y^4)^3} = 0$

(b) We get $\lim_{x\to 0} \frac{0}{(x^2)^3} = 0$

(c) We get $\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{x^4 \cdot x^2}{(x^2+x^2)^3} = \lim_{x\to 0} \frac{x^6}{(2x^2)^3}$

$= \lim_{x\to 0} \frac{x^6}{8x^6} = \frac{1}{8}$

(d) We do not have uniqueness of limits. So, the limit does not exist.

(8) **Fill-in-the-blank [based on Colley]** Suppose that y is defined implicitly as a function $y(x)$ by an equation of the form $F(x, y) = 0$. (So y cannot readily be solved for y in terms of x.) Assume that F and $y(x)$ are both differentiable.

Let us view x and y as functions of t, where $x = x(t) = t$ and $y = y(t)$. Since $F(x, y) = 0$ we know that $F_t(x, y) = 0$. We know that

$$0 = \frac{dF}{dt} = F_x(x, y)\frac{dx}{dt} + F_y(x, y)\frac{dy}{dt}.$$

The second of these equalities follows by the Chain Rule.
As $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{dy}{dx}$, we can express $\frac{dy}{dx}$ as the following

$0 = F_x(x,y) \cdot 1 + F_y(x,y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-F_x(x,y)}{F_y(x,y)}$.

What is striking/interesting about this result? (Two sentence maximum response.)

We can find an expression for $\frac{dy}{dx}$ despite y being implicitly defined.

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$