MULTIVARIABLE CALCULUS
EXAM 1
FALL 2018

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. A formula sheet is on the last page. Best of luck.

(1) Determine the distance between the point $P_0 = (2, 2, 4)$ and the line $\ell(t) = t(1, 1, 1) + (4, 0, 0)$.

Date: October 10, 2018.

(2) Consider the graph of

$$f(x, y) = z = x^2 + y^2 + 2xy$$

and the point $(1, 1, 4)$.

(a) Give an equation for the tangent plane at the given point.
(b) Is the function differentiable at this point? Why?
(c) What is the direction of steepest ascent at this point on this graph?

(3) In what vector space does the graph of $g(x, y, z) = x^2 + y^2$ "live"?

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Describe or sketch the level *surfaces* of this graph when $c = 1, 4$ and 9.

(4) (a) State the Cauchy-Schwarz Inequality.
(b) State the Triangle Inequality.
(c) Give a pair of vectors in $\mathbb{R}^2$ that show equality can be achieved in the Cauchy-Schwarz Inequality while simultaneously **not** achieving equality in the Triangle Inequality. Show the calculations that support your claim.

(5) [Problem from O. Knill]

- Give a formula $f(x, y)$ for the distance in the plane from a point (x, y) to the unit circle centered at the origin. (Hint: you may wish to consider two cases.)
- Show that it satisfies the Eiconal equation $f_x^2 + f_y^2 = 1$.

(6)
- What is the cross product of the two vectors $\mathbf{a} = (1,1,1)$ and $\mathbf{b} = (2,5,2)$?
- What are the spherical coordinates of $(x,y,z) = (1,\sqrt{3},2)$?

(7) Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y^4}{(x^2+y^4)^3}$$

(a) Evaluate the limit along the line $x = 0$.
(b) Evaluate the limit along the line $y = 0$.
(c) Evaluate the limit along the parabola $x = y^2$.
(d) What can you conclude about this limit?

(8) **Fill-in-the-blank [based on Colley]** Suppose that y is defined implicitly as a function $y(x)$ by an equation of the form $F(x, y) = 0$. (So y cannot readily be solved for y in terms of x.) Assume that F and $y(x)$ are both differentiable.

Let us view x and y as functions of t, where $x = x(t) = t$ and $y = y(t)$. Since $F(x, y) = 0$ we know that $F_t(x, y) = 0$. We know that

$$0 = \frac{dF}{dt} = F_x(x, y)\frac{dx}{dt} + F_y(x, y)\frac{dy}{dt}.$$

The second of these equalities follows by ________________________________.
As $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{dy}{dx}$, we can express $\frac{dy}{dx}$ as the following

________________________________.

What is striking/interesting about this result? (Two sentence maximum response.)

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2 + y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$