MULTIVARIABLE CALCULUS
EXAM 1
FALL 2014

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. Best of luck.

(1) [5 points] Give a set of parametric equations for the line through $(5, 0, 6)$ that is perpendicular to the plane $2x - 3y + 5z = -1$.

Note that a normal vector to the plane is $\vec{n} = 2\vec{i} - 3\vec{j} + 5\vec{k}$. The line must be parallel to $\vec{n}$.

So, $\vec{l}(t) = (-5i + 0j + 6k) + t(2\vec{i} - 3\vec{j} + 5\vec{k})$

or $\vec{l}(t) = (5, 0, 6) + t(2, -3, 5)$

Thus, a set of parametric equations is

$$x = 5 + 2t$$
$$y = -3t$$
$$z = 6 + 5t$$

Date: October 9, 2014.

(Sorry(!) for my initial typo!)

(2) [5 points] Find a value for A so that the planes $8x - 6y + 9Az = 6$ and $Ax + y + 2z = 3$ are parallel.

For the planes to be parallel, the coefficients of the first $(8, -6, 9A)$ ought to be a scalar multiple of the second $(A, 1, 2)$, i.e. find the k for which $(8, -6, 9A) = k(A, 1, 2)$.

We see from the 2nd entries that $-6 = k$. So, from this and the first entry $8 = -6A$, which implies $A = -\frac{4}{3}$.

(3) [5 points] Find the distance between the two planes from the previous problem.

We need to find the distance between $\Pi_1: 8x - 6y - 12z = 6$ and $\Pi_2: -\frac{4}{3}x + y + 2z = 3$.

Let us find a point P_1 on Π_1: $(0, 0, -1/2)$.

Let us find a point P_2 on Π_2: $(0, 0, 3/2)$.

Thus $\overrightarrow{P_1P_2} = (0, 0, 2)$

The normal vector to Π_1 is $\vec{n} = 8\vec{i} - 6\vec{j} - 12\vec{k}$, $\vec{n} = (8, -6, -12)$.

We calculate $\text{proj}_{\vec{n}} \overrightarrow{P_1P_2} = \dfrac{\vec{n} \cdot \overrightarrow{P_1P_2}}{\vec{n} \cdot \vec{n}} \vec{n} = \dfrac{-24}{244}(8, -6, -12)$.

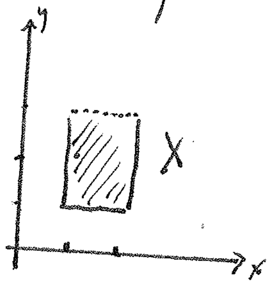
Thus, the distance between the planes is

$$D = \left\| \frac{-24}{244}(8, -6, -12) \right\| \quad \text{- and}$$

$$= \frac{+24}{244}\sqrt{244}$$

(4) [5 points] Briefly explain why the following set is not open: $X = \{(x, y) : 1 \leq x \leq 2, 1 \leq y < 3\}$.

A useful figure



The set is not open because it contains at least one of its boundary points. For example, the point $(1,1)$ is on the boundary since any open ball centered at $(1,1)$ contains some points not in X.

(5) [5 points] State what it means for a scalar-valued function to be *smooth*.

A scalar-valued function is said to be smooth if it has continuous partial derivatives of all orders.

(6) [5 points] A function $f(x, y)$ has $f(1, 2) = 3$ and $\lim_{(x,y)\to(1,2)} f(x, y) = 4$. Does this tell you anything about the differentiability at $(1, 2)$? What? Justify your answer.

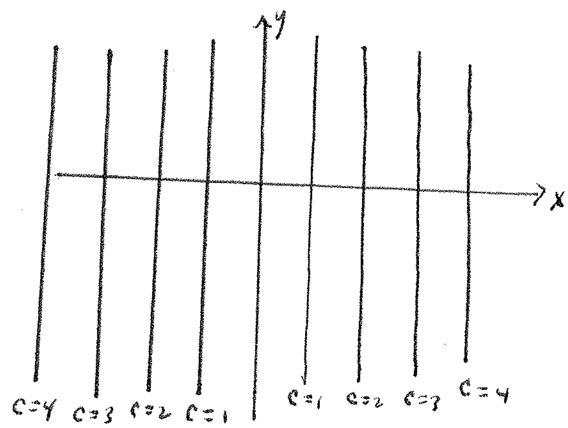
Yes, it tells us first that the function is not continuous since $f(1,2) \neq \lim_{(x,y)\to(1,2)} f(x,y)$.

Now recall Theorem 3.6 (on page 122), which states that, if f is differentiable at (a,b), then it is continuous at (a,b).

As the function here is not continuous, we know it fails to be differentiable.

(7) [10 points] Determine (and draw) several – say, at least four – level curves of the given function f (and make sure to indicate the height c of each curve).

$$f(x,y) = |x|$$



If $c=1$, then $1=|x| \iff x=\pm 1$.

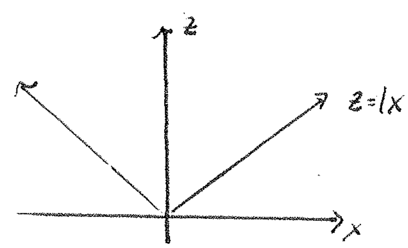
If $c=2$, then $2=|x| \iff x=\pm 2$.

$\vdots$

and so on.

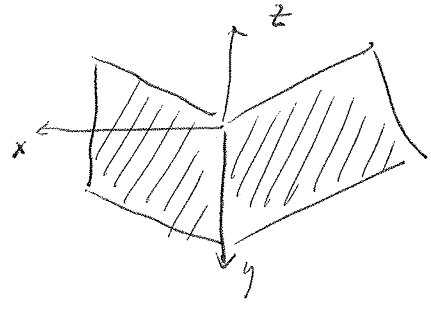
Now give (and draw) four sections of the graph of f by planes of the form $y=c$, where c is a constant.

For any constant c, we still have $z=|x|$.
Thus any one such section looks like another.
Each such looks like



Use the above information to sketch the graph of f or describe it accurately in words.

The graph of f looks like a piece of paper with a fold in it, with this fold sitting on the y-axis.



[5 points]
(8) Find an equation for the hyperplane tangent to the 4-dimensional paraboloid $x_5 = 10 - (x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2)$ at the point $(2, -1, 1, 3, -8)$.

The tangent hyperplane at $(\vec{a}, f(\vec{a}))$ is given by $x_5 = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$.

Thus, we must find the gradient vector $\nabla f(\vec{x})$.

$$\nabla f(\vec{x}) = (-2x_1, -6x_2, -4x_3, -2x_4)$$

And so

$$\nabla f(2, -1, 1, 3) = (-4, 6, -4, -6)$$

Thus,

$$x_5 = -8 + (-4, 6, -4, -6) \cdot (x_1 - 2, x_2 + 1, x_3 - 1, x_4 - 3)$$

$$x_5 = -8 - 4(x_1 - 2) + 6(x_2 + 1) - 4(x_3 - 1) - 6(x_4 - 3)$$

$$x_5 = -8 - 4x_1 + 8 + 6x_2 + 6 - 4x_3 + 4 - 6x_4 + 18$$

$$x_5 = -4x_1 + 6x_2 - 4x_3 - 6x_4 + 28.$$

(9) [5 points] Let $\mathbf{g}: \mathbb{R}^3 \to \mathbb{R}^2$ be a differentiable functions such that $\mathbf{g}(1,-1,3) = (2,5)$ and $D\mathbf{g}(1,-1,3) = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}$ Suppose that $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is ~~de- nied~~ defined by $\mathbf{f}(x,y) = (2xy, 3x - y + 5)$. What is $D(\mathbf{f} \circ \mathbf{g})(1,-1,3)$?

We know by the Chain Rule that

$D(\vec{f} \circ \vec{g})(1,-1,3)$ equals $D\vec{f}(\vec{g}(1,-1,3))D(\vec{g}(1,-1,3))$

We are given $D(\vec{g}(1,-1,3))$. We need to find $D\vec{f}(\vec{g}(1,-1,3))$.

So, $D\vec{f} = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}$

Thus $D\vec{f}(\vec{g}(1,-1,3)) = D\vec{f}(2,5)$

by the given: $g(1,-1,3) = (2,5)$

$= \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}.$

So, we get

$$D(\vec{f} \circ \vec{g})(1,-1,3) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix}$$

[5 points]

(10) Consider a surface with height $f(x, y) = 10 - x^2 - y^2$,

- Find the direction to head if starting at the point $(1, 2, 5)$ if you wish to experience steepest descent.

To head in the direction of steepest descent, we head in the direction of $-\nabla f(1,2)$.

$$\nabla f = (-2x, -2y)$$

And so we head in the direction given by

$$-(-2, -4) = (2, 4).$$

- Now find the path of steepest descent starting from this point.

The path of steepest descent is the curve in xy-plane which is always tangent to the direction of steepest descent of z.

For the curve $y(x)$ to be tangent to $-\nabla f$, its slope must equal the rise over the run of the negative of the 2d gradient vector:

$$\frac{dy}{dx} = \frac{+2y}{+2x} = \frac{y}{x}$$

This is a separable differential equation, which we now solve:

$$\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + C$$ and exponentiating

$$\Rightarrow |y| = C|x|$$

$$y = Cx$$

when $x = 1$, $y = 2$, thus $C = 2$

$$\Rightarrow y = 2x.$$

Note: A sketch of the surface of f would confirm this answer is reasonable.

(11) [5 points] Use the Cauchy-Schwarz inequality to prove that for real numbers $d_1, \ldots, d_n$ we always have

$$d_1^2 + \ldots + d_n^2 \geq \frac{(d_1 + \ldots + d_n)^2}{n}$$

.

The Cauchy-Schwarz inequality states that for all vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$, we have $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$.

Let's rewrite the above inequality in order to make a better parallel to the CS-inequality:

$$n(d_1^2 + \ldots + d_n^2) \geq (d_1 + \ldots + d_n)^2 \qquad (*)$$

(There was no problem with cross-multiplying here since n is always positive.)

With the choice of $\vec{d} = (d_1, \ldots, d_n)$ and $\vec{1} = (1, \ldots, 1)$, ~~applied~~ via an application of the CS inequality, we get

$$|d_1 + d_2 + \ldots + d_n| \leq \sqrt{d_1^2 + \ldots + d_n^2} \cdot \sqrt{1 + \ldots + 1}$$

$$|d_1 + \ldots + d_n| \leq \sqrt{d_1^2 + \ldots + d_n^2}\,\sqrt{n}$$

We now square both sides to obtain $(*)$.