MULTIVARIABLE CALCULUS
EXAM 1
FALL 2013

Name: Answer Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

55 points total.

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes, texts and collaboration are not permitted. Best of luck.

(1) [5 points] Find the angle between the following two non-zero vectors: $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{i} + \mathbf{k}$.

We use Theorem 3.3 of Section 1.3, which states that for $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ we have $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

So we perform the following computations:

$$\vec{a} \cdot \vec{b} = (2,1,-3)\cdot(1,0,1) = -1$$
$$\|\vec{a}\| = \sqrt{4+1+9} = \sqrt{14}$$
$$\|\vec{b}\| = \sqrt{1+1} = \sqrt{2}$$

$\Rightarrow$ $-1 = \sqrt{14}\cdot\sqrt{2}\cos\theta$

and so

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{28}}\right)$$

(2) [5 points] State the Triangle Inequality. Indicate under what conditions equality holds.

As found on page 51 of text:

For all vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, we have $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

Equality holds under the ~~same~~ condition that $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|$. (This is seen by examining our proof of the statement, which relied on the Cauchy-Schwarz Inequality.) When this happens we see that the angle θ between $\vec{a}$ and $\vec{b}$ is 0.

Date: October 10, 2013.

(3) [10 points] Give an equation for the plane containing the following three non-collinear points $(1,0,0),(0,5,0),(0,0,2)$. Let the points be P_0, P_1, P_2, respectively.

We first find two vectors in the plane:

$$\overrightarrow{P_0P_1} = (-1,5,0), \quad \overrightarrow{P_0P_2} = (-1,0,2).$$

Next we find a normal to the plane by computing the cross-product of these two vectors

$$\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 10\vec{i} + 2\vec{j} + 5\vec{k} = \vec{n}$$

Now recall that the equation for a plane is given by $\vec{n} \cdot \overrightarrow{P_0P} = 0$

So, we obtain $(10\vec{i} + 2\vec{j} + 5\vec{k}) \cdot ((x-1)\vec{i} + y\vec{j} + z\vec{k}) = 0$

$$10(x-1) + 2y + 5z = 0 \quad \text{or} \quad 10x + 2y + 5z = 10$$

Now give the set of parametric equations for this plane.

We may use Proposition 5.1 on page 44, which states that for a plane Π containing $P_0(\vec{c})$ and parallel to nonzero, nonparallel vectors $\vec{a}$ and $\vec{b}$ is $\vec{X}(s,t) = s\vec{a} + t\vec{b} + \vec{c}$

Thus, with $\vec{a} = \overrightarrow{P_0P_1}$ and $\vec{b} = \overrightarrow{P_0P_2}$, we get

$$\vec{X}(s,t) = s(-\vec{i} + 5\vec{j}) + t(-\vec{i} + 2\vec{k}) + (1,0,0)$$

$$= (1-s-t)\vec{i} + (5s)\vec{j} + (2t)\vec{k}$$

The individual parametric equations are thus:

$$x = 1-s-t$$

$$y = 5s$$

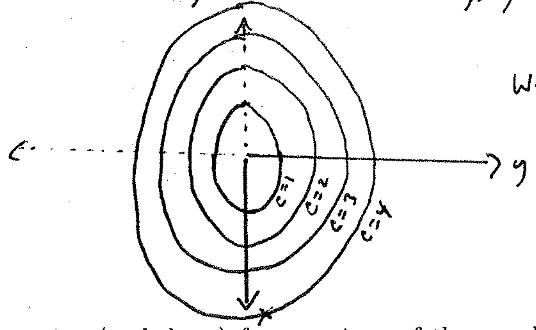
$$z = 2t$$

See page 88 of the text.

(4) [10 points] Determine (and draw) several – say, at least four – level curves of the given function f (and make sure to indicate the height c of each curve).

$$f(x, y) = 4x^2 + 9y^2$$

For any choice of positive c, the equation $c = 4x^2 + 9y^2$ is that of an ellipse. Note that $4x^2+9y^2 \geq 0$ for all (x,y), so there are no level curves of negative height.

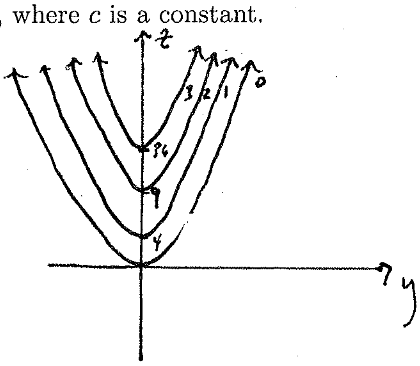


We have concentric ellipses, with levels increasing as we move outwards.

Now give (and draw) four sections of the graph of f by planes of the form $x = c$, where c is a constant.

We project these sections onto the yz-plane.

We see parabolas, with vertex height increasing as we move away from the origin.



If $x=0$, then $z = 9y^2$

$x=\pm 1$ then $z = 4+9y^2$

$x=\pm 2$ then $z = 16+9y^2$

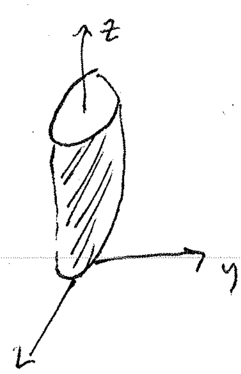
$x=\pm 3$ then $z = 36+9y^2$

We also notice symmetry across the yz-plane. (Note that sections of the type $y=c$ would yield something similar.)

Use the above information to sketch the graph of f.

Putting this together we get the following:

(a bowl with horizontal "cuts" that are ellipses and vertical "cuts" are parabolas, and "sitting" on the origin.)



We have the graph of an elliptic paraboloid, as in Figure 2.24 on page 94.

(5) [5 points] Use a change of coordinates to polar coordinates to find the following limit. (Recall that $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$ are the polar to cartesian and cartesian to polar change of coordinates, respectively.) **Hint:** $-1 \le \cos(\theta) \le 1$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + x^5}{x^2 + y^2}$$

First we make a change of coordinates.

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + r^5\cos^5\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

By our Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$, we may re-write the denominator of the quotient.

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + r^5\cos^5\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta + r^3\cos^5\theta.$$

$$= \lim_{r\to 0} r \cdot \lim_{r\to 0} \cos^3\theta + r^2\cos^5\theta$$

The first limit is obviously 0, and to show the second is finite we take advantage of the hint. The hint implies that

$$-1 - r^2 \le \cos^3\theta + r^2\cos^5\theta \le 1 + r^2,$$ and so the limit of the second is between -1 and 1 as $r\to 0$.

This shows that the limit we seek is 0.

This is Example 10 on page 105.

(6) [10 points] Find an equation for the plane tangent to the graph of $f(x,y) = 4\cos(xy)$ at the point $\mathbf{a} = (\pi/3, 1, 2)$.

We may use the equation of the tangent plane found on page 121 of the text.

$$h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So,

$$f_x = -4\sin(xy)\cdot y \qquad f_x(\pi/3, 1) = -4\cdot\frac{\sqrt{3}}{2}\cdot 1 = -2\sqrt{3} \qquad \text{and } f(\tfrac{\pi}{3}, 1) = 2$$

$$f_y = -4\sin(xy)\cdot x \qquad f_y(\pi/3, 1) = -4\cdot\frac{\sqrt{3}}{2}\cdot\frac{\pi}{3} = -\frac{2\pi\sqrt{3}}{3}$$

Thus,

$$h(x,y) = 2 - 2\sqrt{3}\left(x - \frac{\pi}{3}\right) - \frac{2\pi\sqrt{3}}{3}(y-1)$$

is the equation for the tangent plane.

This tangent plane is a good linear approximation of f near $\mathbf{a}$ only if f is differentiable at $\mathbf{a}$. Since the existence of the above plane doesn't guarantee differentiability, how can you be sure that f is differentiable at $\mathbf{a}$?

By Theorem 3.5 (or more generally Theorem 3.10), as the first-order partial derivatives exist and are continuous - each is the product of the sine function and a polynomial - then f is differentiable at the point, and so the above is indeed the tangent plane and a good linear approximation of f near the point of tangency.

(7) [5 points] Calculate $D(f \circ \mathbf{g})$ in two ways: (a) by first evaluating $f \circ \mathbf{g}$ and (b) by using the chain rule and the derivative matrices Df and $D\mathbf{g}$.

$$f(x,y) = x^2 - 3y^2, \;\; \mathbf{g}(s,t) = (st, s+t^2)$$

(a) First we compute $f \circ \vec{g} = (st)^2 - 3(s+t^2)^2$

$$= s^2t^2 - 3(s^2 + 2st^2 + t^4)$$

$$= s^2t^2 - 3s^2 - 6st^2 - 3t^4$$

And so $D(f \circ \vec{g}) = \begin{bmatrix} 2st^2 - 6s - 6t^2 & 2s^2t - 12st - 12t^3 \end{bmatrix}$

(b) Now using the chain rule:

$$Df = \begin{bmatrix} 2x & -6y \end{bmatrix} = \begin{bmatrix} 2st & -6(s+t^2) \end{bmatrix}$$

and $D\vec{g} = \begin{bmatrix} t & s \\ 1 & 2t \end{bmatrix}$ and thus

$$Df\,D\vec{g} = \begin{bmatrix} 2st & -6(s+t^2) \end{bmatrix} \begin{bmatrix} t & s \\ 1 & 2t \end{bmatrix}$$

$$= \begin{bmatrix} 2st^2 - 6s - 6t^2 & 2s^2t - 12st - 12t^3 \end{bmatrix}$$

(8) [5 points] Suppose that the four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ in $\mathbb{R}^3$ are coplanar (i.e., they all lie in the same plane). Show that then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$.

The vector $(\vec{a} \times \vec{b})$ is perpendicular to both $\vec{a}$ and $\vec{b}$, by definition of cross product, and so is orthogonal to the plane in which they sit. A similar statement holds for the vector $(\vec{c} \times \vec{d})$. Thus, the vectors $(\vec{a} \times \vec{b})$ and $(\vec{c} \times \vec{d})$ are parallel, and so the angle between them is either 0 or π. This makes the length of their cross product 0, and so their cross product is $\vec{0}$.

A purely computational approach to this problem is also possible – i.e. by computing first $(\vec{a} \times \vec{b})$, then $(\vec{c} \times \vec{d})$ and then the product of these – but this is tedious!