GAME THEORY
ECON 380
FALL 2007
Office Hours: Tues. & Thurs. 4-5pm or by appointment

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OVERVIEW

The aim of *game theory* is to investigate and understand the manner in which rational people should interact when they have complimentary or conflicting interests. So far, game theory only provides answers in simple situations, but these answers have already led to a fundamental restructuring of the way economic theorists think about the world. And, the time is not too distant in the future when the same will be true for all the social sciences.

In this class we will examine the theory of games in detail. In the first half of the class we study the basics of the theory including what a game is and how simple games are solved. In the second half of the class we examine more complicated games and applications. Along the way we may also study some experiments that have been conducted to test the ability of game theory to predict the outcome of interactions between real (and perhaps not rational) people.

The book we will use for the course is available at the bookstore. It is: *Games for Business and Economics, 2nd edition* by Roy Gardner.

OPPORTUNITY COSTS (my one rant)

Let's get one thing straight at the beginning – my job is not to minimize your opportunity costs. The extracurricular activities (e.g., athletics) that you participate in take time and may cause conflicts with your classes. In other words, there are *time opportunity costs to participating in these activities.* I have designed the course to minimize any potential conflicts (midterms will happen in class and the entire class schedule is posted on the back) but your choices may impact your performance in this class. The opportunity costs implied by your choices are yours, not mine. For example, I do repeat lectures if you miss class and I do not give make up exams.

GRADING

Your performance in the class will depend on three things: Homework Assignments, Midterms, and a Final Exam. Midterm exams are not cumulative, but the final exam is. This means that you will need to study all the topics we cover in the class for the final exam. The in-class exam dates are written on the class schedule (see the back of this sheet) and are not negotiable. However, people may get sick and miss an exam. With this in mind, only the best 2 of the 3 midterms will count towards your grade.

Everything you do for this course will be graded on a 100 point scale. Your final grade for the course will be determined according to the percentages listed below and your performance relative to the performance of the class as a whole.

$$\textit{Your class grade} = (0.1\times \textit{Homework})+(0.3\times \textit{MT1 Grade})+(0.3\times \textit{MT2 Grade})+(0.3\times \textit{Final Grade})$$

After returning each exam, I will provide you with a grade distribution for the class. I will assign individual grades relative to the overall performance of the class. In general, I look for natural break points in the range of grades, but the average on any given component of the class will usually be close to a B.

On the back of this sheet you will find a schedule of topics (book chapters) for the course.

Schedule of Topics

Week	Dates	Homework due	Topic	Book Chapter
1	9/11,13		Introduction to Game theory & Chance	1, 2
2	9/18	#1 (on 18th)	Nash Equilibrium for Two-Person Games	3
2	**9/20**		**No Class (Professor giving lecture at Laval)**	-
3	9/25,27	#2 (on 27th)	Mixed Strategies	4
4	10/2	#3 (on 2nd)	Catch up and Review	-
4	**10/4**		**First Midterm Exam (in class)**	1-4
5	10/9,11		n-Person Games	5
6	10/16,18	#4 (on 18th)	Market Structure	6
7	10/25	#5 (on 25th)	Credibility & Subgame Perfection	7
8	**10/30**		**Second Midterm Exam (in class)**	5&6
8	11/1		Credibility & Subgame Perfection (cont.)	7
9	11/6,8	#6 (on 8th)	Repeated Games	8
10	11/13,15	#7 (on 15th)	Evolutionary Game Theory	9
11	**11/20**	#8 (on 19th)	**Third Midterm Exam (in class)**	7-9
12	11/27,29		Signaling Games and Sequential Equilibrium	10
13	12/4,6	#9 (on 6th)	Games Between a Principal and an Agent	11
14	**t.b.a.**	#10 (on 10th)	**Final exam**	1-11

Game Theory	Jeffrey Carpenter
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Problem set number 1	jpc@middlebury.edu
Due: Tuesday September 18th (in class)	x3241

Please answer all the following questions and try to write legibly.

1) Book Chapter 1 problems: 2, 3, 4, 10.

2) The landlord and the eviction notice:
A landlord has three tenants, A, B, and C, in a rent-controlled apartment building in NYC. A new law says that the landlord has the right to evict one tenant per building. The landlord calculates that the values of a vacant apartment is $15,000, both to the tenant and to her. She send the following letter to each of the tenants: "Tomorrow I will visit you building. I will offer A $1000 if he agrees to vacate his apartment voluntarily; otherwise, I will evict him. If A agrees to vacate, I will offer $1000 to B, and if she refuses, I will evict her. If she accepts, I will evict C."

   (a) Write a game tree for this situation, and find the dominant strategy equilibrium.
   (b) What if there were 10 tenants instead of three?
   (c) What if the landlord offers $10 instead of $1000.

Note: For the next two questions you can consult you book. They are mentioned in chapters 1 and 12.

3) The second-price auction:
A single object is to be sold at auction. There are n>1 bidders, each submitting a single bid, in secret, to the seller. The value of the object to bidder i is v_i. The winner of the object is the highest bidder, but the winner only pays the second highest bid.

   (a) Show that "truth-telling" (i.e., each player bids v_i) is a weakly dominant strategy for each player. Assume, where convenient, there are no ties.
   (b) Does the analysis depend on whether the other bidders tell the truth too?
   (c) Can you speculate about why real bidders might not behave this way?

4) What is the dominant strategy in the English or ascending price auction (this is the standard auction mechanism you typically think of)? Assume that bidder i bids b_i and has a private value of v_i.

5) Recall that one can find equilibria by the method of iterative deletion of strictly dominated strategies. Consider the following game: In a football game, the offense has two strategies, Run or Pas. The defense has three strategies: Counter Run, Counter Pass or Blitz. The expected payoff (in yards gained) to each combination appears below. Use elimination of dominated strategies to find an equilibrium in this game (list the order in which you eliminate strategies).

		Defense		
		Counter Run	Counter Pass	Blitz
Offense	Run	3, -3	7, -7	15, -15
	Pass	9, -9	8, -8	10, -10

6) Book Chapter 2 problems: 2, 5, 7, 12.

7) You and a rival are engaged in a game in which there are three possible outcomes: you win, your rival wins, and the two of you tie. You get the payoff of 50 if you win, 20 if you tie and 0 if you lose. What is your expected payoff in the following situations where nature decides if you win or not:

   (a) There is a 50% chance the game ends in a tie, 10% chance you win (and therefore a 40% chance you lose).
   (b) There is a 50-50 chance of winning and there are no ties.
   (c) There is an 80% chance you lose and a 10% chance you win or tie.

8) Now, say you are indifferent between the three strategies (regardless of their payoffs) and will therefore choose a strategy randomly. What is your expected payoff from playing randomly?

9) Reconsider (3). Now calculate your expected utility if your utility function is U=ln(payoff).

Game Theory
Econ 380
Problem set number 2
Due: Thursday September 27th (in class)

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Please answer all the following questions and try to write legibly.

1) Book Chapter 3 problems: 1, 2, 5, 7, 8, 9.

2) *Rock, Paper, Scissors* is a symmetric two-player, zero-sum game. The rules are simple Rock beats (i.e. smashes) Scissors, Scissors beats (i.e. cuts) Paper, and Paper beats (i.e. covers) Rock.

   (a) Write down a normal form payoff matrix for this game with 1 indicating a win, 0 a tie, and -1 a loss.
   (b) Identify what a *maximin/minimax* player would do.
   (c) Identify any pure strategy Nash equilibria.
   (d) How would you play this game?

3) Country A and Country I are at war. The two countries are separated by a series of rivers, illustrated in the figure below.

```
A ~~~ b ~~~ c
|     |     |
d ~~~ e ~~~ f
|     |     |
g ~~~ h ~~~ I
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Country I sends its navy with just enough supplies to reach A. The fleet must stop for the night at intersections (e.g. if the fleet takes the path IhebA, it must stop for the first night at h, the second night at e and the third at b). if unhindered, on the fourth day the fleet will reach A and destroy country A. Country A can send its own navy to prevent this. Country A's fleet has enough supplies to visit three contiguous intersections, starting from A (e.g. Abcf). If A's fleet catches I's fleet (i.e. if both navies stop for the night at the same place), it destroys the fleet and wins the war.

(a) List the strategies for the two countries (say A can go backwards and wants to be at d or b on the last night – otherwise the matrix is huge) and make a normal form payoff matrix (assume the winner gets 1 and the loser -1).
(b) Eliminate strictly dominated strategies. What happens to the matrix in (a)?
(c) Eliminate **weakly** dominated strategies. Note that the order matters when you eliminate weakly dominated strategies. Write down the two possible resulting matrices.
(d) Identify any Nash equilibria.

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Due: Tuesday October 2nd (in class) | x3241

Please answer all the following questions and try to write legibly.

1) Book Chapter 4 problems: 1, 2, 3, 4, 5, 7, 10.

2) Correlated Equilibrium: Consider the Up-Down/Left-Right game played by Alphonse and Gaston, with normal form shown below.

	G Left	G Right
A Up	5,1	0,0
A Down	4,4	1,5

(a) Find all the equilibria. And the expected payoffs for both players at each equilibrium.
(b) Show that the two can achieve average payoffs of (3,3) as a Nash equilibrium if they are given a fair die.

3) Hawk-Dove-Bourgeois: Consider the following game description. There are two types in the population, those called Hawks because they are very aggressive, and those called Doves because they avoid confrontation. Hawks and Doves are mixed in the population and randomly interact in pairs. They interact to either share of fight over a prize of value, V. When two Hawks meet, they fight over the prize and each has an equal chance of winning. The cost of fighting *and losing* is C ($V<C$). When two Doves meet, they share the prize equally without fighting. When a Hawk meets a Dove, the Dove gets scared and runs away. In this case the Hawk gets the whole prize and the Dove gets nothing.

(a) Write down the normal form of this game.
(b) Find all the Nash equilibria.
(c) Now consider the addition of a third strategy, Retaliator. Retaliators behave like a Dove against another Retaliator, but, if its opponent escalates, Retaliators escalate too and act like Hawks. Last, Retaliators can bully Doves and therefore do a little better when meeting them. Assume that Retaliators get ¾ of the value in an interaction with a dove, but the two do not fight. Write down the new normal form.
(d) Find all the equilibria of the new normal form (set V=1 and C=2).
(e) Using Symmetry, Efficiency and Payoff dominance to see which equilibria seem more likely.

4) Modeling Team Production: Suppose there are *n+1* workers who work as a team to produce output that they share equally. Workers have t_i (i=1…$n+1$) hours to spend either working or relaxing and each individual, *i*, chooses how many hours to work. Call this choice x_i. Workers value leisure at a constant rate of α utils and working provides output which workers value at q. For reasons that will become obvious below, assume $q/(n+1)<\alpha<q$.

(a) What is each worker's objective function? Label it π_i (hints: each worker receives a 1/(n+1) share of the total output from the team so what is team output? The remaining hours are worth α each.)
(b) How many hours does each worker work if they make identical choices? (hint: this should solve the problem max π_i by choosing x_i.)
(c) Show that this is a social dilemma. (hint: show that if they all work fully they all do better)

5) Modeling Team Production with Reciprocal Agents (this is hard): Now let's redo our analysis of team production making it both more and less complicated. To complicate things let's say that there is some likelihood that workers punish shirkers (i.e. those who don't work hard). We also complicate things by making effort costly. Working costs the worker b. However, to simplify things say working or shirking is now a binary choice and all workers face the exact same incentives so we can loose the index, i.. That is, we still have n+1 workers but now let $e \in \{0,1\}$ be the discrete choice of each worker to work or shirk. Each workers objective function can now be formalized as,

$$\pi = \frac{(1-\sigma)(1+n)q}{(1+n)} - be$$

where (1-σ) is the probability that any team member works. The condition $(1-\sigma)q < b < q$ assures this remains a social dilemma. Now consider the case of the $1+n^{\text{th}}$ team member.

(a) When this member decides to shirk, she imposes a cost of q on the team. Therefore, what is the *net benefit of shirking*? (hint: this is the benefit of shirking minus the benefit to working)

(b) Now assume workers reciprocate the cost imposed by shirkers on the group by punishing them. Let m be the probability that any member of the team monitors. Further let s be the punishment that monitors can impose on shirkers. If there are n team members, this means the expected punishment the $1+n^{\text{th}}$ team member can expect is nms. Including the expected punishment, what is the new net benefit of shirking?

(c) What is the equilibrium frequency of monitoring, m^*?

(d) Define, p as the "psychic" payoff people get from catching and punishing shirkers. Define the net benefit to monitoring as the benefit from monitoring another person minus the benefit from not monitoring. If 1-σ is the frequency of workers, then σ is the frequency of shirkers in the group. Say monitors get a benefit of p per shirker caught, but have to pay a cost, c, to monitor someone else. Write down the net benefit to monitoring.

(e) What is the equilibrium probability of anyone shirking, σ^*?

(f) Now say p is a function of the size of the group or $p(n)$ where $p'(n)<0$ so people get less benefit from punishing in larger groups (maybe because the groups are more anonymous). Formally prove that large groups will suffer from more shirking than samll groups?

Game Theory	Jeffrey Carpenter
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Problem set number 4	jpc@middlebury.edu
Due: Thursday October 18th (in class)	x3241

Please answer all the following questions and try to write legibly.

1) Book Chapter 5 problems: 1, 4, 6, 7, 8 and 10.

2) Here's a three player sequential and simultaneous game. Say 3 department stores (A, B and C) consider opening new stores in one of two malls. The urban mall is smaller and can only accommodate 2 stores, but the rural mall is larger and can handle 3 stores. Each store prefers to be in a mall with one or more other stores over being alone in a mall because malls with multiple stores attract more customers and profits will be higher. Further, stores prefer the urban to the rural mall because urban customers have more money to spend. Each store must choose between trying to get space in the Urban mall and in the Rural mall. Say the stores rank the five possible outcomes as follows: 5, (the best) Urban mall with one other store; 4, rural mall with one or two other stores; 3, alone in urban mall; 2, alone in rural mall; and 1, (worst) alone in rural mall after getting rejected in the urban mall.

   (a1) Sequential move game - let's first assume the stores move in the following order: A then B then C. Draw the extensive form game and fill in the payoffs (A,B,C) at the terminal nodes (assume A gets screwed if too many choose U).

   (a2) List the strategies for the three stores based on the extensive form in (a1). There should be 16 strategies for C, 4 for B, and 2 for A.

   (a3) Convert the extensive form from (a1) into a normal form game. I suggest creating a matrix for B and C and then allow A to choose the matrix.

   (a4) Find any pure strategy Nash equilibria. in the two versions of the game.

   (b1) Simultaneous move game – now let's say the game is played simultaneously so that A, B, and C don't know what each other have done when making a decision. Draw a normal form for this game. It should be different from what you got in (a3).

   (b2) Find any pure or mixed strategy equilibria in this game.

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Problem set number 5 | jpc@middlebury.edu
Due: Thursday October 25th (in class) | x3241

Please answer all the following questions and try to write legibly. There are NO book problems this time.

1) Market demand is given by p=140-q. There are two firms, each with unit costs = $20. Firms can choose any quantity. Find the Cournot equilibrium and compare it to the monopoly outcome and the perfect competition outcome. Why aren't the latter equilibria of the market game?
2) Suppose that in problem 1, firm 1's unit costs fall to $10, while firm 2's don't change. This gives firm 1 a cost advantage. How much does firm 1 produce at the Cournot equilibrium. Is this more than firm 2? Why?
3) Suppose that market demand is nonlinear, taking the form $p=100-q^2$. Suppose that firms' unit costs are $5. Find the Cournot equilibrium. What does this suggest about the properties of Cournot competition in general?
4) Prove the Cournot Limit Theorem for the following markets: (a) market demand is p=80-4q, and each firm has unit cost = $10; (b) the market in problem 3.
5) Suppose that two firms both have average variable costs = $50. Market demand is q=100-p. Find the Bertrand equilibrium. Would your answer change if there were three firms?
6) Same demand as in problem 5. Find the Bertrand equilibrium if firm 1 has average variable cost = $40 and firm 2 has average variable cost = $60. Show that the Bertrand equilibrium changes when firm 2 has average variable cost = $90. Explain.
7) Same demand as in problem 5, only now there are two firms with average variable cost = $50 and two firms with average variable cost = $60. Find the Bertrand equilibrium. Is it the same as the perfectly competitive equilibrium?
8) Here is a model of cigarette product differentiation. There are n brands of cigarettes. Each brand of cigarettes, i, has the demand curve:

$$x_i = 15,000 - 1000 p_i - (1000n)(p_i - \textit{average price})$$

where p_i is brand *i's* price in dollars. Average cost is $1/pack throughout the industry. What happens to industry profits at Bertrand equilibrium when the number of firms rises from $n=2$ to $n=3$? How does this relate to the price of Marlboros?
9) Suppose there are two countries, labeled 1 and 2. let x_i be the tariff level of country i (in percent), for i=1,2. If country i picks x_i and the other country (j) picks x_j, then country i gets a payoff of $2000 + 60x_i + x_i x_j - x_i^2 - 90x_j$ (measured in billions of dollars). Assume that x_1 and x_2 must be between 0 and 100 and that the countries set tariff levels simultaneously.

    (a) Find the best response functions for the two countries.
    (b) Compute the Nash equilibrium.
    (c) Show that the countries would be better off it they made a binding agreement to set lower tariffs (than in equilibrium).
    (d) Graph the best response functions to show the equilibrium and the social optimum.
10) Redo number 1 finding the von Stackelberg or Leader-follower equilibrium. Say firm one is the leader.

Game Theory | Jeffrey Carpenter
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Problem set number 6 | jpc@middlebury.edu
Due: Thursday November 8th (in my mailbox) | x3241

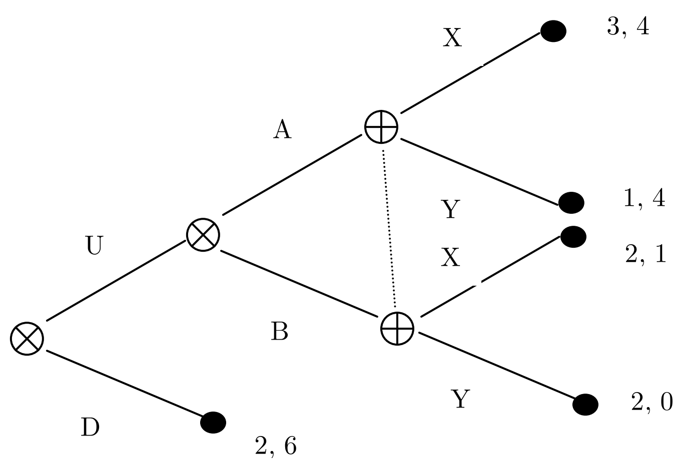
Please answer all the following questions and try to write legibly.

1) Book Chapter 7 problems: 1, 2, 3, 4, 5, 7.

2) In book problem 7, suppose that firm 2, which moves second, has unit cost = c. What value must c be in order for firm 2 to have the same market share as Firm 1 in the Stackelberg equilibrium? This cost advantage is a measure of how big the first mover advantage is.

3) Consider the following game in extensive form. The players are called Cross and Plus. There is an information set at Plus' move.

   (a) Write down the normal form of this game. Note: Cross' strategies have two components and Plus' strategies have only one component.
   (b) Find the pure strategy equilibria in the normal form.
   (c) What is the subgame perfect equilibrium?



Game Theory	Jeffrey Carpenter
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Problem set number 7	jpc@middlebury.edu
Due: Thursday November 15^{th} (in class)	x3241

Please answer all the following questions and try to write legibly.

1) Show that the 2-person Cournot market game in which p=130-Q, AC=10, and $Q=q_1+q_2$ played twice has a unique subgame perfect (SGP) equilibrium. Also show that the 2-person game below played three times, has a unique SGP equilibrium.

	C	D
C	3,3	0,4
D	4,0	2,2

2) Find the average payoffs of the most socially efficient subgame perfect equilibria for the following game played twice.

	E	S
E	-50,-50	100,0
S	0, 100	0,0

3) Find three strategies that yield an average payoff of (2.5, 2.5) in the following game played twice.

	A	B
A	3,3	1,4
B	4,1	0,0

4) Find a SGP equilibrium that supports monopoly payoffs for the Cournot market game played an infinite number of times where p=130-Q, $Q=q_1+q_2$, and AC=30. Say both firms have discount factors of 0.90.

5) Suppose in problem (4) that firm 1 has AC=10 and firm 2 has AC=30. Find the one-shot Cournot equilibrium. What are the profit possibilities of this game repeated infinitely often (you need to read the book for this one).

6) Consider the following game:

		Player 2		
		X	Y	Z
	X	3, 3	0, 0	0, 0
Player 1	Y	0, 0	5, 5	9, 0
	Z	0, 0	0, 9	8, 8

(a) What are the equilibria of the stage game?
(b) If the players could write a contract to play a strategy that could be costlessly enforced by a court, what would they agree to do?
(c) Suppose, however, they can't write a contract. If the game is played for 2 periods, show there is a subgame perfect equilibrium in which {Z, Z} is played in the first period.

7) Consider the following "war of attrition" game. Interaction between players 1 and 2 takes place over discrete periods of time, starting in period 1. In each period, players choose between "stop" (S) and "continue" (C) and they receive payoffs given by the following stage game matrix:

		Player 2 S	Player 2 C
Player 1	S	x, x	0, 10
	C	10, 0	-1, -1

However, the length of the game depends on the players' behavior. Specifically, if one or both players selects S in a period, then the game ends at the end of this period. Otherwise, the game continues into the next period. Suppose the players discount future payoffs according to the discount factor $0<d<1$. Assume $x<10$.

(a) Assume players use mixed strategies. Show that, if $d>(11-x)/(12-x)$ the infinitely repeated version of this game has an equilibrium in which players use the mixed strategy forever. But if the inequality goes the other way, the players end the war of attrition today.

Game Theory | Jeffrey Carpenter
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Problem set number 8 | jpc@middlebury.edu
Due: Monday November 19th (in my mailbox) | x3241

Please answer all the following questions and try to write legibly.

Chapter 9 Evolutionary Game Theory

1) Find an ESS and draw a phase diagram for the Prisoner's Dilemma with payoffs below.

	C	D
C	3,3	0,4
D	4,0	2,2

2) Find two ESSs and draw a phase diagram for the following coordination game. Which equilibrium has a larger basin of attraction?

	A	B
A	3,3	0,0
B	0,0	2,2

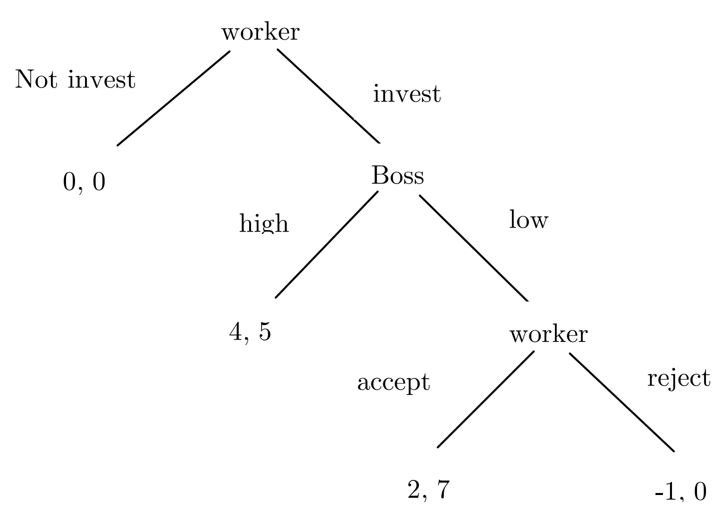
3) Find the ESSs and draw a phase diagram for the following Hawk-Dove game.

	A	B
A	-49,-49	2,0
B	0,2	1,1

4) Find any ESSs and draw a phase diagram for the following asymmetric game.

	O_2	L_2
O_1	6,4	5,5
L_1	9,1	10,0

5) Consider the following "hold-up" game expressed in extensive form:



Where workers invest in human capital or not, then bosses offer high wages or low wages and then workers accept the low wage or reject it. Note, this model implies that if the worker invests she can generate 10 units of profit in the firm, but it costs her 1 unit to invest. Further, we assume high wages are always accepted.

(a) Write down the normal form of this game when you consider the two strategies of the boss (High or Low) and only three strategies for the worker (not invest, invest accept, and invest reject).
(b) Is there a "hold-up" problem here? Why or why not.
(c) Construct a phase diagram for this game based on the replicator dynamic (note: there are two populations).
(d) Which, if any, of the equilibria are ESSs?
(e) Now assume a fraction d of the players in each population mess up and choose randomly while (1-d) of the players choose based on the replicator dynamic. What are the critical values of d such that the ESS(s) in (d) are still stable.

Game Theory
Econ 380
Problem set number 9
Due: Thursday December 6th (in class)

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Please answer all the following questions and try to write legibly.

1) Book Chapter 10 problems: 1, 2, 6.

2) Consider the following game with asymmetric information. The open circle is Nature's initial move, closed circles are for the informed first mover and open boxes are for the uniformed second mover. Further, probabilities are listed in brackets.

    (a) Fully describe a *separating* equilibrium in this game, if one exists.
    (b) Fully describe a *pooling* equilibrium in this game, if one exists.