Teaching Portfolio
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A. Teaching Philosophy

As an undergraduate, the realisation that mathematics is a creative subject captivated my attention. I discovered that the technical rules I had learned in high school were formulated in a rich mathematical landscape. By engaging with theorems and abstract definitions, I began to understand what it was I had been doing in my previous mathematics classes.

When I began to teach mathematics my approach to instruction was to try to share this experience with my students: if I could enable students to grasp the bigger picture then I believed their mastery of the subject would be greatly enhanced. Initial student feedback indicated that many students were struggling to connect the mathematical ideas I presented with an effective approach to problem-solving. It was apparent that students needed to build their own understanding rather than adopting my knowledge. I realised that being an effective educator required a broad and more interactive approach. Since that time my teaching has been continually evolving. Each new teaching role has taught me different aspects of how to engage with students and support their learning experience.

Recently, this has been as a member of Harvard University's multivariable calculus teaching staff. My role is to lead a class of approximately 20 students in both instruction and discussion, simultaneously stimulating engagement with new ideas and methods of problem-solving. Facilitating student learning in this way has been a new experience and an exciting challenge. I have had to think carefully about how to effectively introduce a new concept and allow for students to gain practice problem-solving, leading to a significantly more dynamic classroom environment than I have been used to. To achieve this, I create worksheets that set the narrative for the class, allowing students to discover concepts and formulate results based on their investigations. For example, when introducing the directional derivative, students worked on a problem that guided them towards a formulation of the relationship between the numerical value of a directional derivative at a point and the direction in which a function is increasing most rapidly. In doing so, students discovered a connection between the directional derivative and the (familiar) single variable derivative. As a result, students commented that they now understood why this connection should exist.

As my teaching methods have matured, my teaching ideals have come to be focused on three principles: ownership, accessibility, and engagement. My main teaching goal is to uphold these principles in a comfortable learning environment wherein students can actively participate, and demonstrate and measure their progress. I shall briefly expand upon these principles below.
Ownership: I am dedicated to my students’ development as individual learners. I want students to leave a lecture or discussion beginning to form their own intuition for a new concept and to have the confidence to share their insights with others. I have a strong belief that actively discussing mathematics enables student learning and I provide many opportunities for students to do so. In discussion sections, I regularly use group worksheets to achieve this, and problems are generally investigative: for a proof-based course, students are asked to ‘prove or give a counterexample’.

Adopting this approach has stimulated student participation and increased student involvement in resolving difficulties. As a teaching assistant at the University of California, Berkeley, this was made apparent to me during an office hour discussion of dual bases: a group of students were struggling with the basic definition of the dual basis of a given basis. After reinterpreting the textbook definition, the students were successfully able to determine how to write a specific given linear functional as a linear combination of a dual basis. My function in this situation was to lead students to the more tractable definition of the dual basis and to facilitate their engagement with this concept.

Accessibility: As a teacher and mentor of mathematics students, I encounter a wide variety of backgrounds and abilities. Being a relatable and approachable person is an important part of fostering a learning environment that caters to all students. Whether I am supervising a single student or leading a discussion, students must feel comfortable discussing their work with me. I aim to provide sufficient opportunity for students to do so, regularly holding office hours, encouraging students to contact me with any queries that arise, and holding extensive review sessions. An important part of being an accessible teacher is being able to clearly and appropriately respond to student queries. When a student asks me a question, I believe that it is my responsibility to patiently direct students towards a satisfactory answer. This requires revisiting and reinterpreting topics for students frequently. I am mindful not to make students feel intimidated by their difficulties and will often recall my own troubles learning a subject. I try to supply students with a path towards a solution rather than stating it outright. For example, when I’ve been the instructor for an upper-division linear algebra course, a student may ask ‘Can you explain why this homomorphism is injective?’, and I’ll respond ‘Well, what does it mean for a function to be injective? What theorems can you use to show a homomorphism is injective?’.

Having a mutual dialogue with a student about their difficulties allows them to recognise suitable approaches to a solution.

Engagement: In the classroom I aim to make the subject approachable and tractable. This allows students to grasp the mathematics that they are learning ‘in action’. In all of my teaching roles, encouraging an interactive and open dialogue with students is fundamental to this. I aim to regularly interact with students during class, looking to engage them with the material in an informal manner. For example, before introducing the abstract definition of a ring in an upper-division abstract algebra course I will ask a class if they can think of five examples of sets equipped with notions of addition and multiplication. This exercise sets the narrative for the lecture and engages students to consider unfamiliar concepts of addition and multiplication. Once an abstract notion has been introduced, I will highlight typical (and more subtle) features with appropriate examples. I will reinforce an idea through a variety of worksheet problems, asking students to find (counter-)examples or to formulate their own generalisations of a theorem.

Teaching mathematics has strengthened my passion for the subject greatly. Sharing and guiding the learning experience of students has forced me to re-evaluate the subtleties and difficulties that can arise in the subject. As my teaching roles have expanded, I have had to reevaluate how to effectively engage students with concepts and problem-solving techniques. I have endeavored to make learning mathematics a reachable goal, and to be a teacher that students can relate to comfortably. As I continue my teaching career, I look forward to reinforcing these foundations and adapting to the challenges ahead.

B. Teaching & Mentoring Responsibilities

I have fulfilled a variety of teaching and mentoring responsibilities at Harvard University and the University of California, Berkeley (UCB). I have taught undergraduate (lower- and upper-division) and graduate level courses as an instructor, led discussion sections as a teaching assistant, supervised undergraduate individual reading programs as a graduate mentor, and supervised advanced undergraduates in programs of original summer research.
B1. Teaching Responsibilities

As a course instructor I am solely responsible for course design and content, lecturing material, constructing and administering exams, issuing homework and quizzes, leading discussion sections, and holding office hours.

I have been the instructor for the following courses at Harvard University:

  Upper-division undergraduate mathematics course.
- **Canonical Bases and Geometry (Math 284)**, Spring 2017.
  Graduate-level ‘Topics in Mathematics’ course.

I have been the instructor for the following courses at the University of California, Berkeley:

- **Linear Algebra & Differential Equations (Math 54)**, Summer 2010, 2011.
  Lower-division mathematics course, approx. 40 students per course.
- **Linear Algebra (Math 110)**, Summer 2012.
  Upper-division mathematics course, approx. 30 students.
- **Abstract Algebra (Math 113)**, Summer 2014.
  Upper-division mathematics course, approx. 30 students.

In addition, I have been a Teaching Fellow at Harvard University, a position that combines the role of instructor and discussion leader. I lead small sections in both instruction and discussion, being responsible for delivery of course content, facilitating interactive discussion and problem-solving, constructing worksheets, and holding office hours. I have been a Teaching Fellow for the following courses at Harvard University:

- **Multivariable Calculus (Math 21a)**, Fall 2016.
  Lower-division undergraduate mathematics course (two sections), approx. 20 students per section.

I served as a graduate student instructor (GSI) at the University of California, Berkeley, for a variety of lower- and upper-division mathematics courses:

- **Calculus (Math 1B)**, **Multivariable Calculus (Math 53)**, **Linear Algebra & Differential Equations (Math 54)**, **Linear Algebra (Math 110)**, **Abstract Algebra (Math 113)**, **Graduate Algebra (Math 250A)**.

As a GSI I provide assistance and support to professor-led lectures. I am responsible for leading discussion sections 1-3 times per week, assessing student work, holding office hours and review sessions, and providing academic support to students. In this role I have encountered a broad range of students (~700) students over nine semesters) from a multitude of academic and social backgrounds.

B2. Mentoring Responsibilities

In the summer of 2015 I organised an 8-week research opportunity for undergraduates at the University of California, Berkeley, similar to the Research Experience for Undergraduates (REU) program. The participants were advanced undergraduate students who were coming to Berkeley from across the US. I supervised a group of six students, overseeing research projects on toric degenerations of (partial) flag varieties and computing string cones. Preparation and background was provided through daily lectures and problem sessions. During the latter stage of the program I was responsible for providing academic support and encouragement in a research environment. In addition, I arranged for a special lecture series by faculty from the Mathematics Department. For further details see Section F.
The Directed Reading Program (DRP) pairs undergraduates at the University of California, Berkeley, with a graduate mentor for a semester long program of supervised individual learning. I participated in this program in each academic semester between Fall 2014 - Spring 2016, supervising reading programs on:

- **representation theory of the symmetric group**;
- **Hilbert’s Nullstellensatz and algebraic geometry**;
- **representation theory of finite dimensional Lie algebras**;
- **representation theory of quivers and the Ringel-Hall algebra**.

My function is to provide a reading schedule to students, usually following an advanced undergraduate/beginning graduate level textbook, or a peer-reviewed research article. I supplement students’ reading with worksheets and problem sets, and provide academic support with weekly one-on-one meetings. At the end of the semester I prepare students for a 10-minute presentation of their work, offering advice on presentation skills and timekeeping.

### C. Teaching Methods

In the past, I have primarily taught a course by lecturing and aim to keep my lecturing style approachable and well-presented. I am mindful of the standard of my boardwork and keep my lecturing pace regulated by writing in large, capital letters at all times. I stop frequently to summarise an argument or remind students where the lecture is heading, looking to break down the barrier between lecturer and audience by regularly asking for questions.

In my most recent teaching experience at Harvard University, leading a multivariable calculus class of approximately 20 students, I have approached the classroom differently. The lower-division calculus courses at Harvard are focused on an interactive approach to teaching, combining both discussion and lecture into a more dynamic classroom environment. As a result, I have made use of worksheets to set the narrative for the class, allowing students to learn new concepts while simultaneously gaining practice at problem-solving (see Appendix 3). Teaching in this way requires careful consideration of appropriate exercises that students can work through, while simultaneously introducing and/or reinforcing concepts.

To determine how best to proceed in a teaching role, it is vital to understand the academic background of your students. Prior to the first classroom interaction, I distribute an anonymous online survey focusing on student background and expectations for the course. For example, in an upper-division Abstract Algebra summer course I asked students: *Why were they taking the course? What exposure to upper-division mathematics did they have? Had they written formal proofs before?* The responses allow me to determine how much time I should invest on certain preliminary topics, as well as create appropriate homework sets at the beginning of the course.

In all teaching situations, I outline my expectations and goals for the program of study with a concise syllabus (see Appendix 1). I place a strong emphasis on the importance of student participation and provide suggestions for increased student engagement. I indicate the amount of time I expect students to spend on the course and provide study tips. I like to remind students that learning mathematics is challenging but that hard work and perseverance will lead to positive results.

It is important that students are presented with a well-structured, concise narrative, placing new topics in context with earlier material. At the beginning of a course, I issue a proposed outline for the daily program of topics to be discussed. Each week I provide a complete set of self-contained, typed lecture notes for the upcoming classes so that students are aware of that week’s focus. I highlight aspects of a new definition or result with a variety of clear examples and accentuate key points with brief, eye-catching phrases (see Appendix 2). Any notes that I provide are made available at a frequently updated course website. Student have responded positively to the notes that I provide, and my lecture notes have been used as a supplementary reference by a Professor at UC Santa Cruz (see Appendix 8).

As a GSI at the University of California, Berkeley, I would engage students’ understanding of a new idea using investigative worksheets. Leading a discussion section, I will give students the opportunity to engage with problems in small groups (say 3-4 students). I have found that students are more willing to engage with their peers in active
discussion when confronted with open-ended problems. This approach to problem-solving is also highly effective when supervising advanced undergraduates: one of my students investigating the representation theory of Lie algebras correctly (re)discovered a basic result concerning tensor products of irreducible representations of sl2. This experience has led to a marked increase in his understanding and confidence with the material. Examples of worksheets are found in Appendix 4, 5.

In all of my teaching interactions, I take the time to place results in a wider mathematical context and to illuminate the interactions between different branches of mathematics. This provides motivation for a subject and allows students to gain perspective on what they are learning. Several of my students have pursued topics that were first discussed in a class I taught or during office hours through subsequent independent programs of study with me. Some have gone on to participate in prestigious national summer undergraduate research opportunities.

D. Representative Course Materials

It is important to be well-prepared in any teaching or mentoring role. Designing a course is a considerable challenge and requires me to think about the overall course structure well in advance. The same preparation must be applied to all distributed material and it is essential that anything provided to students is clearly presented. In the Appendices I provide representative examples of the following material:

- the course syllabus for Fall 2016 Multivariable Calculus course at Harvard University (Appendix 1);
- notes from a lecture on Lagrange’s Theorem (Appendix 2);
- a worksheet from a Fall 2016 Multivariable Calculus class at Harvard University (Appendix 3);
- a worksheet from a Fall 2015 Linear Algebra discussion section at the University of California, Berkeley (Appendix 4);
- a worksheet from an undergraduate individual reading program in Fall 2015 on the representation theory of the symmetric group (Appendix 5);
- a final exam from my Summer 2012 Linear Algebra course at the University of California, Berkeley (Appendix 6).

E. Evaluation of Teaching Effectiveness

Interacting with students in a variety of teaching and mentoring roles has presented many challenges for me. It is important that I am able to determine if I am engaging the students effectively. Obtaining student feedback through anonymous evaluations provides a measure of my effectiveness as a teacher and focuses my efforts to improve my teaching.

At the University of California, Berkeley, students have the opportunity to provide numerical evaluations and comments. These comments are useful in providing a sharper focus of how students respond to my teaching efforts. Below I provide a representative sample of student comments and numerical evaluations from recent years.

In recognition of my ‘outstanding work in teaching undergraduates’, I was awarded an Outstanding Graduate Student Instructor Award by the University of California, Berkeley, in 2013 (see Appendix 7).

Student comments:

- “George is an amazing teacher! He explains concepts with precision and clarity. The worksheets are extremely useful!”
  Directed Reading Program student, University of California, Berkeley.
- “Always comes to class very, very prepared; puts in an incredible amount of effort. George is extremely helpful and is always willing to answer questions in office hours, via e-mail, through Piazza, and in class. One of the best teachers I have ever had.”
  Math 110 student Summer 2012, University of California, Berkeley.

- “You have a real talent for teaching, especially when it comes to understanding and relating to difficulties your students are having. You are able to explain things in a way that is very approachable and I can see you've put a lot of work into doing your job well.”
  Math 110 student Fall 2015, University of California, Berkeley.

**Numerical Evaluations:**
Evaluations scored on a 7 point scale: 1 (not at all effective) - 7 (extremely effective)

**Course Instructor:**
At the University of California, Berkeley, course instructors are evaluated on two criteria: the effectiveness of their teaching, and the overall course.

<table>
<thead>
<tr>
<th>Course</th>
<th>Semester</th>
<th>No. of responses</th>
<th>Effectiveness</th>
<th>Overall</th>
<th>Ave.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math 54</td>
<td>Summer 2010</td>
<td>28</td>
<td>6.2</td>
<td>5.9</td>
<td>6.05</td>
</tr>
<tr>
<td>Math 54</td>
<td>Summer 2011</td>
<td>36</td>
<td>6.3</td>
<td>6.2</td>
<td>6.25</td>
</tr>
<tr>
<td>Math 110</td>
<td>Summer 2012</td>
<td>19</td>
<td>6.5</td>
<td>6.4</td>
<td>6.45</td>
</tr>
<tr>
<td>Math 113</td>
<td>Summer 2014</td>
<td>18</td>
<td>6.4</td>
<td>6.4</td>
<td>6.4</td>
</tr>
</tbody>
</table>

**Course GSI:**
These are a sample of my teaching effectiveness evaluations as a GSI. I include evaluations from my first teaching experience in Fall 2009. The marked increase and consistency in my later student evaluation scores provide evidence in support of my improvement as a teacher.

<table>
<thead>
<tr>
<th>Course</th>
<th>Semester</th>
<th>No. of responses</th>
<th>Effectiveness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math 1B</td>
<td>Fall 2009</td>
<td>52</td>
<td>5.75</td>
</tr>
<tr>
<td>Math 54</td>
<td>Fall 2012</td>
<td>53</td>
<td>6.7</td>
</tr>
<tr>
<td>Math 110</td>
<td>Fall 2013</td>
<td>20</td>
<td>6.25</td>
</tr>
<tr>
<td>Math 110</td>
<td>Spring 2014</td>
<td>34</td>
<td>6.65</td>
</tr>
<tr>
<td>Math 110</td>
<td>Fall 2014</td>
<td>20</td>
<td>6.5</td>
</tr>
</tbody>
</table>
F. Undergraduate Research

It can be difficult for undergraduate students to appreciate the process that research mathematicians undertake to ask questions that lead to interesting theorems. This is especially true in areas of pure mathematics, where no obvious physical situation may be generating mathematical problems.

Summer research opportunities allow exceptional undergraduates to experience what it is like to be a mathematician. However, it is a challenge to provide students with accessible problems that generate interesting results. Student backgrounds must be taken into account when devising research projects so that all participants are engaged and can gain a sense of achievement.

In 2014 I was asked by the ‘Geometry & Topology’ Research Training Group at UCB to oversee the organisation of an 8-week summer research opportunity for undergraduates from across the US. The program’s aim is to provide exceptional undergraduates with research experience in geometry and topology, preparing prospective graduate students for research.

In addition to organising practical matters (national advertisement, participant selection, student housing, program finances), I supervised a group of six students on a program of research relating to toric degenerations of flag varieties. I worked with my PhD advisor to design suitable research projects that would be accessible to the participants. Some students wrote software packages to test conjectures while others considered theoretical problems. Providing different types of projects ensured students could engage with a research problem that aligned with their academic background.

Having the opportunity to work closely with undergraduates on research projects was a fantastic experience. Devising suitable problems that would be accessible to talented undergraduates was a new and exciting challenge. I am firmly committed to providing similar research opportunities in pure mathematics in the future.

G. Professional Development

Engaging with students has become a major aspect of my passion for mathematics. It is important to me that I provide students with an outstanding education and that they are provided with ample opportunity to further themselves academically.

I have taken advantage of several workshops and courses in an effort to improve my teaching:

- In Spring 2015, I successfully completed the graduate course ‘Mentoring in Higher Education’ at the University of California, Berkeley.
- I have attended the following workshops offered through the University of California, Berkeley, Teaching & Resource Center: ‘How Students Learn’, ‘Fostering Student Participation’.

These courses and workshops have been extremely valuable in improving my approach to effective teaching. For example, I have utilised ‘mentoring contracts’ with students undertaking individual programs of study. This idea was introduced to me in the Mentoring in Higher Education course and has been a valuable way to coherently set student and mentor expectations at the beginning of a program of study.

In the future I will continue to make use of teaching improvement opportunities. In light of my teaching experience at Harvard this past semester, I am very interested to learn how to effectively apply Inquiry Based Learning techniques in the mathematics classroom.

H. Conclusion

Teaching mathematics has been a tremendously enriching experience. My teaching and mentoring roles have given me a strong appreciation of the challenges facing students as they undertake programs of study in mathematics. Progressing from my position as a graduate student at the University of California, Berkeley, to a Lecturer at Harvard
University has affirmed the great responsibility that college educators undertake to guide students towards their academic goals. It has been important to me to be a personable and relatable teacher that students can trust with their difficulties and I look forward to sharing in future students’ education. Having the opportunity to positively affect the student learning experience is an exciting prospect, and one that I am committed to realising in the years to come.

I. Appendices

- **Appendix 1**: Multivariable Calculus Fall 2016 course syllabus
- **Appendix 2**: Lecture on Lagrange’s Theorem. Abstract Algebra, Summer 2014
- **Appendix 3**: Multivariable Calculus Fall 2016 worksheet
- **Appendix 4**: Linear Algebra Fall 2015 discussion worksheet
- **Appendix 5**: Directed Reading Program Fall 2014 worksheet
- **Appendix 6**: Linear Algebra Summer 2012 exam
- **Appendix 7**: UC Berkeley GSI Award
- **Appendix 8**: email from UCSC Professor
Appendix 1

TuTh 10-11.30am, Science Center 110.

Instructor: George Melvin
Contact: gmelvin@math.harvard.edu
Office: Science Center 209h
Office Hours: M 9-10am, Tu 5-6pm, W 10-11am. All office hours will be held in SC209.
Class Webpage: http://math.harvard.edu/~gmelvin/teaching.html
Worksheets, handouts, and any other information presented during class can be found here.
Course Assistant: Wentong Zhang (wentongzhang@college.harvard.edu)
Weekly Problem Session: TBA

Homework: Homework is collected at the beginning of every class. Homework must be submitted in hardcopy by 10.20am. Late homework will not be accepted. If you are unable to submit homework on time, due to legitimate circumstances, then let me know (via email) ASAP.

You are free to discuss solutions to homework sets with your peers; in fact, this is strongly encouraged! However, you must write up your homework independently. If you have worked with someone else then write ‘I worked with I. Newton, C.F. Gauss,...’ on your submission. You should not make use of any online forums (eg math.stackexchange.com); if you have questions then there are many resources available to you (see Resources below). You are reminded of the Honor Code pertaining to your work.

Homework should be legible and written in complete English sentences.

What you should expect in class: Class will start promptly at 10.07am. Class will consist of a mixture of lecture, discussion, and problem-solving. The aim is to provide you with the tools and confidence to attack your homework sets. We will frequently skim or omit basic concepts that you can read about on your own (see Resources below), or discuss with your peers outside of class. Familiarise yourself, ahead of time, with topics in the syllabus by glancing through one or more of the resources.

How to succeed: This class will be fast-paced and it is extremely important to keep up with the material. You should attend section and expect to be spending considerable time outside of class thinking about what we have discussed; it would not be uncommon to spend 10-15hrs per week on this course at various stages. If you feel like you are struggling then come and see me as soon as possible; it is my job to help. It is extremely important to ask questions, no matter how silly you might think they are. I guarantee that others in the class will have similar silly questions. You are not expected to know everything straight away, but you are expected to be given the opportunity to learn.

Here are some tips:
- form study groups and learn from each other,
- ask lots of questions - to your peers, to the instructors,
- get started on your homework as soon as possible.

Community Rules: You should feel comfortable in class, both with your peers and instructors. In order to maintain an encouraging and comfortable environment for all, please keep the following in mind:

1. Be inclusive - if someone is struggling, help them. Learning is not a competition! Explaining mathematical concepts to others is the best way to solidify your understanding.
2. Don’t trivialise difficulties - refrain from saying ‘Oh, that’s easy’.
3. No interrupting.

Resources: The following references should be considered your immediate points of contact if you have questions:

- Math 21A instructors, CAs - you are free attend all instructors’ office hours.
- Math Question Center - Sun-Thu 8.30-10.30pm, SC309. Ask questions, find answers!
- Oliver Knill’s Course Webpage - http://sites.fas.harvard.edu/~math21a/

The following resources are a great supplement to the material we discuss in class:
- MIT OpenCourseWare - video of Prof. Auroux’s Fall ’07 Multivariable Calculus course.
- UC Berkeley - Prof. E. Frenkel’s Fall ’09 Multivariable Calculus course is available on youtube.com.
- Online notes - recommended links are available at the Class Webpage.

There are thousands of multivariable calculus resources available online. I recommend finding notes or a textbook that you like and maintaining a ‘vow of celibacy’ to them: the material we cover is standard fare, and jumping around between resources is more of a hinderance than an aid to your learning experience.

Finally: Remember that maths is hard so don’t feel ‘stupid’, we all have to start somewhere. If you put in the effort then I guarantee that you will be rewarded. 😊
Keywords: Lagrange's Theorem. Groups of prime power are cyclic. Cyclic group, group generator.

In today's lecture we will prove one of the most fundamental results in group theory - Lagrange's Theorem. This result states that there are strong conditions on the existence of subgroups of a finite group. Moreover, we will classify all groups \( G \) such that \(|G| = p\) is prime. We will also introduce a class of groups - the cyclic groups - and completely describe their structure; not bad for a day's work!

Remark. For the remainder of the course we will suppress the '∗' when considering the law of composition in a group. As such, we will simply say 'Let \( G \) be a group', where the law of composition is implicitly understood to have been defined as part of the definition of \( G \).

5.1 Lagrange's Theorem

Theorem 5.1.1 (Lagrange's Theorem). Let \( G \) be a finite group, \( H \subset G \) a subgroup. Then, the number of left cosets of \( H \) in \( G \) is \(|G|/|H|\). That is,

\[
|G/H| = |G|/|H|.
\]

Hence, the order of \( H \) divides the order of \( G \).

Proof: Since \( G \) is finite then \( H \subset G \) is finite and, by Corollary 4.2.3, every left coset of \( H \) in \( G \) has the same size, equal to \( k = |H| \). Since a left coset of \( H \) in \( G \) is an equivalence class of the equivalence relation \( \sim_H \) we know that there is a partition of \( G \) into equivalence classes. If there are \( r \) such equivalence classes (ie \( r \) left cosets of \( H \)), each of which has the same size \( k \), then

\[
|G| = k + \cdots + k = rk \quad \implies \quad |G/H| = r = |G|/|H|.
\]

Corollary 5.1.2. Let \( G \) be a finite group, \( g \in G \). Then, \( o(g) \) divides \(|G|\).

Proof: Recall that \( o(g) = |\langle g \rangle| \), and \( \langle g \rangle \subset G \) is a subgroup of \( G \). The result follows from Lagrange's Theorem.

Hence, the order of an element \( g \), \( o(g) \), divides the order of \( G \).

Example 5.1.3. 1. Suppose that \( H \subset D_8 \) is a subgroup. Then, \(|H|\) must be even. Indeed, Lagrange's Theorem implies that \(|H|\) divides \(|D_8| = 8\), we must have \(|H| = 1, 2, 4, 8\). Note that Lagrange's Theorem does not imply that there must exist a subgroup of each of these orders. We will come back to this problem when we discuss Sylow's Theorems.

2. Let \( f : \mathbb{Z}/5\mathbb{Z} \to S_4 \) be a group homomorphism. Then, \( f \) must be the trivial homomorphism. Indeed, since \( \ker f \subset \mathbb{Z}/5\mathbb{Z} \) is a subgroup then \(|\ker f| = 1, 5\); if \(|\ker f| = 1\) then \( f \) is injective. Hence, there must exist an element of \( S_5 \) of order 5, but 5 does not divide \(|S_4| = 24\).

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1 Why do we make an issue of this? It could be possible to define two different laws of composition on a set \( G \) so that we obtain two different groups \((G, \cdot)\) and \((G, \bullet)\) with the same underlying set.
Remark 5.1.4. Why have we repeatedly used the adjective ‘left’? There is an analagous notion of a right coset of \( H \) in \( G \): define an equivalence relation on \( G \) by
\[
g \sim^H g' \iff g'g^{-1} \in H.
\]
It can be shown that this defines an equivalence relation on \( G \) and the equivalence classes are of the form \([g] = \{ hg | h \in H \} \) defined \( Hg \).

The resulting partition of \( G \) is called the right \( H \)-partition of \( G \), and we denote the set of equivalence classes \( H \setminus G \).

There are analogus results to those obtained above for right cosets of \( H \) in \( G \) - in particular, there is an analogue of Lagrange’s Theorem - the number of right cosets of \( H \) in \( G \) equals \(|G|/|H|\) - so that, for a finite group \( G \) and a subgroup \( H \subset G \)

5.2 Groups of prime order

Let \( G \) be a group of prime order, so that \(|G| = p\) is a prime. Let \( g \in G \) be nontrivial. Thus, the subgroup \( H = \langle g \rangle \) is a nontrivial subgroup of order \( o(g) \) so that \( H = G \), by Corollary 5.1.2. Hence, we have
\[
G = \{ e_G, g, g^2, \ldots, g^{p-1} \}.
\]
Moreover, \( g^p = e_G \), for any nontrivial \( g \in G \).

If \( G' \) is another group of order \( p \) then, for any nontrivial \( h \in G' \), we find that
\[
G' = \{ e_{G'}, h, h^2, \ldots, h^{p-1} \}.
\]
It can then be shown that \( G \) and \( G' \) are isomorphic as groups\(^2\). In particular, any group \( G \) of prime order \( p \) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). Hence, any group of prime order \( p \) is a cyclic group (to be defined in the next section) - a group generated by a single element.

5.3 Cyclic groups

Lagrange’s Theorem is a simple consequence of the existence of a particular equivalence relation that can be defined on any finite group \( G \), given a subgroup \( H \), and has allowed us to classify\(^3\) all finite groups of prime order. Combining Lagrange’s Theorem with the basic arithmetic properties of \( \mathbb{Z} \) from Lecture 2 allows us to to understand the structure of a larger class of groups - the cyclic groups.

Definition 5.3.1 (Cyclic group). A group \( G \) is cyclic if there exists \( x \in G \) such that
\[
G = \langle x \rangle = \{ \ldots, x^{-1}, e_G, x, x^2, \ldots \}.
\]
We call such an \( x \) a generator of \( G \), and say that \( G \) is generated by \( x \).

Remark 5.3.2. Let \( G \) be a cyclic group with generator \( x \in G \). Then, \( x^{-1} \) is also a generator of \( G \). In general, there are many generators of a cyclic group.

Example 5.3.3. a) \( e_G \) is a generator of a cyclic group \( G \) if and only if \( G \) is the trivial group.

b) Let \( G = \mathbb{Z} \). Then, \( G \) is cyclic and generated by 1. The set of all generators of \( G \) is \( \{ \pm 1 \} \).

\(^2\)What is the isomorphism between \( G \) and \( G' \)?

\(^3\)This means that we have understood essentially all finite groups of prime order - they are isomorphic to \( \mathbb{Z}/p\mathbb{Z} \), so that they they have the same structure as \( \mathbb{Z}/p\mathbb{Z} \).
c) Let \( n \in \mathbb{Z}_{>1} \) and \( G = \mathbb{Z}/n\mathbb{Z} \). Then, \( G \) is cyclic and generated by 1. The set of generators of \( G \) is
\[
\{ \overline{x} \in \mathbb{Z}/n\mathbb{Z} \mid x \in \{1, \ldots, n-1\}, \gcd(x, n) = 1 \}.
\]
d) Let \( G = \mu_6 = \{ z \in \mathbb{C} \mid z^6 = 1 \} \), considered as a subgroup of \((\mathbb{C}^\times, \cdot)\), the law of composition being multiplication of complex numbers - \( \mu_6 \) is the group of sixth roots of unity. Then, \( \mu_6 \) is cyclic with generator \( w = \frac{1}{2}(1 + \sqrt{-3}) \).

e) The dihedral group \( D_8 \) is not a cyclic group as there does not exit any element of order 8. In general, \( D_{2n} \) is not cyclic.
f) \( S_n \) is cyclic if and only if \( n = 2 \).
g) \((\mathbb{Q}, +)\) is not cyclic. In fact, the examples above describe all possible cyclic groups:

**Theorem 5.3.4 (Structure Theorem of cyclic groups).** Let \( G \) be a cyclic group generated by \( x \in G \). Then,
\begin{enumerate}
  \item if \( G \) is infinite then \( G \) is isomorphic to \( \mathbb{Z} \);
  \item if \( G \) has order \( n \) then \( G \) is isomorphic to \( \mathbb{Z}/n\mathbb{Z} \).
\end{enumerate}

Let \( H \subset G \) be a nontrivial subgroup. Then,
\begin{enumerate}
  \item \( (G \text{ infinite}) \) \( H \) is isomorphic to \( \mathbb{Z} \) and generated by \( x^i \), where \( i = \min \{ r \in \mathbb{Z}_{>0} \mid x^r \in H \} \).
  \item \( (G \text{ of finite order } n) \) Suppose \( |H| = k \) so that \( n = km \), by Lagrange's Theorem. Then, \( H \) is cyclic and generated by \( x^m \). Hence, \( H \) is isomorphic to \( \mathbb{Z}/k\mathbb{Z} \).
\end{enumerate}

**Proof:**

a) Suppose that \( G \) is infinite and \( G = \langle x \rangle \). Then, by the definition of a cyclic group we have
\[
G = \{ \ldots, x^{-1}, e_G, x, x^2, \ldots, \}.
\]
Define
\[
f : \mathbb{Z} \to \mathbb{Z}; \quad r \mapsto x^r.
\]
Then, \( f \) is a group homomorphism. Moreover, \( f \) is injective - if \( f(r) = e_G \) then \( x^r = e_G = x^0 \) so that \( r = 0 \), by Lemma 3.1.6 - and \( f \) is surjective, as \( x \) is a generator of \( G \). Hence, \( f \) is an isomorphism and \( G \) is isomorphic to \( \mathbb{Z} \).

If \( H \subset G \) is a nontrivial subgroup and \( i = \min \{ r \in \mathbb{Z}_{>0} \mid x^r \in H \} \) then we claim that \( x^i \) generates \( H \): we need only show that any nontrivial \( h \in H \) is of the form \( h = x^{ia} \), for some \( a \in \mathbb{Z} \). So, let \( h \in H \) be nontrivial. Then, we must have \( h = x^r \), for some \( r \in \mathbb{Z} \). By the division algorithm we can find \( q, b \in \mathbb{Z} \) with \( 0 \leq b < i \) such that \( r = qi + b \). Hence, we see that
\[
x^b = x^{r-qi} = x^r (x^i)^{-q} \in H.
\]
Since \( 0 \leq b < i \) and \( i \) is the minimal positive integer such that \( x^i \in H \), we must have that \( b = 0 \) so that \( r = qi \). Hence, \( h = (x^i)^q \) and
\[
H = \langle x^i \rangle = \{ \ldots, x^{-i}, e_G, x^i, x^{2i}, \ldots, \}.
\]

\footnote{This is a particular example of a more general result: any finite subgroup of \((\mathbb{C}^\times, \cdot)\) is cyclic.}
b) If $G$ is finite of order $n$ and cyclic, then

$$G = \langle x \rangle = \{e_G, x, \ldots, x^{n-1}\},$$

and $x^n = e_G$ by Lemma 3.1.6. Define

$$f : \mathbb{Z}/n\mathbb{Z} \to G ; \bar{r} \mapsto x^r.$$

This function is well-defined: if $\bar{r} = \bar{s}$, so that $r - s \in n\mathbb{Z}$, then

$$f(\bar{r}) = x^r = x^{s+nk} = x^s(x^n)^k = x^s(e_G)^k = x^s = f(\bar{s}).$$

Moreover, $f$ is an isomorphism of groups. In a similar way as proved in a), it can be shown that any nontrivial subgroup $H$ is of the stated form.

Example 5.3.5. Let $n = 10 = 2.5$. Then, the subgroups of $\mathbb{Z}/10\mathbb{Z}$ are

$\{0\}, \{0, 5\} \cong \mathbb{Z}/2\mathbb{Z}, \{0, 2, 4, 6, 8\} \cong \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/10\mathbb{Z}.}$
These problems are supplementary; they are not required but they may help you learn.
* problems are related to material that we will see later in the course.

Learning objectives
1. To learn the method of Lagrange multipliers.
2. To apply the method of Lagrange multipliers to constraint problems.
3. To learn sufficient conditions that ensure a solution to global extrema problems.
4. To apply the method of Lagrange multipliers to global extrema problems.

Definitions, formulae, results

- **The Method of Lagrange Multipliers** Let \( f(x, y) \) be a nice function, \( g(x, y) = 0 \) be a curve. The **Lagrange equations** are the following equations in three unknowns \( \lambda, x, y \):

\[
\begin{align*}
    f_x(x, y) &= \lambda g_x(x, y) \\
    f_y(x, y) &= \lambda g_y(x, y) \\
    g(x, y) &= 0
\end{align*}
\]

If \( f(x, y) \) attains a local maximum/minimum on the curve \( g(x, y) = 0 \) at a point \((x_0, y_0)\) then, either

(a) \((\lambda, x_0, y_0)\) is a solution to the Lagrange equations (for some \(\lambda\)), or
(b) \(\nabla g(x_0, y_0) = 0\).

There is an analogous statement for \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = 0 \).

- Solving the Lagrange equations can be hard!

- **Bolzano Theorem** Let \( D \subseteq \mathbb{R}^2 \) be a bounded region that includes all of it’s boundary points, and \( f(x, y) \) be a continuous function defined everywhere on \( D \). Then, \( f(x, y) \) attains a global maximum \( f(x_1, y_1) \) and a global minimum \( f(x_2, y_2) \), at points \((x_1, y_1)\) and \((x_2, y_2)\) in \( D \).

To find the **global maximum/minimum of \( f(x, y) \) on a region \( D \):**

(a) determine the local maxima/minima of \( f(x, y) \) on the interior of \( D \),
(b) determine the maxima/minima of \( f(x, y) \) on the boundary of \( D \) (using Method of Lagrange, for example),
(c) compare the local extrema found above to determine the global extrema.
Extremizing functions on curves

1. (a) Let \( f(x, y) = y \). Does \( f(x, y) \) admit any local maxima/minima in the plane?
(b) Restrict the inputs of \( f(x, y) \) to those \((x, y)\) such that \( x^2 + y^2 = 1 \). Without performing any calculations, where does \( f(x, y) \) attain its maximum? its minimum?
(c) Draw the circle \( x^2 + y^2 = 1 \) and the level curves \( f(x, y) = 1, -1 \). What is the relationship you observe between the level curves, the circle \( x^2 + y^2 = 1 \), and the points you found in (a).
(d) Now, let \( h(x, y) = x^2 - y^2 \). Some level curves of \( h(x, y) \) are drawn below, along with the circle \( x^2 + y^2 = 1 \). Use the plot to determine the points \((x, y)\) on the circle where \( h(x, y) \) is maximised/minimised.

![Plot with level curves and circle](image)

Can you describe a relationship between the points you’ve found, the level curves of \( f(x, y) \) and the curve \( x^2 + y^2 = 1 \)?

(e) Let \( f(x, y) \) be a function, \( g(x, y) = 0 \) a curve in the plane. Based on your investigations above, which of the following statements sounds reasonable? (Several may sound reasonable, or none at all)

i. \( f(x, y) \) is maximised/minimised at the points \((a, b)\) on the curve \( g(x, y) = 0 \), such that the tangent line to \( g(x, y) = 0 \) at \((a, b)\) is parallel to the tangent line (at \((a, b)\)) of some level curve of \( f(x, y) \).

ii. \( f(x, y) \) is maximised/minimised at the points \((a, b)\) such that \( \nabla f(a, b) \) is orthogonal to \( \nabla g(a, b) \).

iii. \( f(x, y) \) is maximised/minimised at the points \((a, b)\) such that \( \nabla f(a, b) = 0 \) and \( \nabla g(a, b) = 0 \).

iv. \( f(x, y) \) is maximised/minimised at the points \((a, b)\) such that \( \nabla f(a, b) \) and \( \nabla g(a, b) \) are parallel.
Lagrange multipliers

2. We are going to solve a Lagrange multiplier problem for \( f(x, y) = 2x - 3y \) on the ellipse \( 4x^2 + y^2 = 10 \).

(a) Write down the Lagrange equations.
(b) Find maxima/minima of \( f(x, y) = 2x - 3y \) on the curve \( 4x^2 + y^2 = 10 \).

3. The earth revolves around the sun on an elliptical trajectory, with the center of the sun as a focus of the ellipse (You don’t need to know what this means). In the celestial plane containing the centres of the earth and the sun, with coordinates \((x, y)\), the ellipse is given by the equation \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \). The centre of the sun is at \((-4, 0)\) in these coordinates. Thus, the distance (squared) of the center of the earth from the center of the sun is \( d(x, y) = (x + 4)^2 + y^2 \). At which points \((a, b)\) is the center of the earth closest to the center of the sun? (Hint: minimising the distance is the same as minimising \( d(x, y) \) )
Global extrema

4. For which regions $D$ is it true that a continuous function $f(x, y)$ must attain a maximum and minimum value on $D$?

(a) $D$ is the set of points $(x, y)$ such that $|x| \leq 1$ and $|y| \leq 3$.

(b) $D$ is the set of points $(x, y)$ such that $-1 \leq x^2 - y^2 \leq 1$. (*The plot in Problem 1 may help*)

(c) $D$ is the set of points such that $4x^2 + y^2 \leq 10$.

(d) $D$ is the set of points such that $|2x - y| \leq 2$.

5. Find the global maximum and minimum of the function $f(x, y) = 3xy - y + 5$ on the disc $x^2 + y^2 \leq 1$. 

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Normal and Self-Adjoint Operators, Spectral Theorem

Throughout this worksheet V will always be a finite dimensional vector space over F = \( \mathbb{R}, \mathbb{C} \). If an inner product is not specified then it will be assumed to be the ‘obvious’ one.

1. a) Give an example of an operator \( T \in \mathbb{L}(\mathbb{C}^2) \) that is not a normal operator. Explain carefully why you know it is not a normal operator.

b) Give an example of a diagonalisable operator \( T \in \mathbb{L}(\mathbb{C}^2) \) that is not normal. Justify your chosen example carefully.

c) Give an example of an operator \( T \in \mathbb{L}(\mathbb{C}^3) \) that is normal but not self-adjoint.

d) Give an example of an operator \( T \in \mathbb{L}(\mathbb{R}^2) \) that is diagonalisable but not self-adjoint.

e) Verify that the operator \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 ; \mathbf{v} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{v} \) is normal. Explain why it’s not self-adjoint.

2. (Longer?) Repeat 1a)-d), replacing ‘\( T \in \mathbb{L}(\mathbb{C}^k) \)’ with ‘\( T \in \mathbb{L}(\mathbb{P}_2(\mathbb{R})) \)’, where \( \mathbb{P}_2(\mathbb{R}) \) admits the inner product \( \langle p, q \rangle = \int_0^1 p(x)q(x)dx \).

3. Let \( (\mathbb{R}^2, \langle , \rangle) \) be the inner product space, with \( \langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2, \mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \).

   a) Define a self-adjoint operator \( T \) on the inner product space \( (\mathbb{R}^2, \langle , \rangle) \) that has eigenvalues \( \sqrt{2}, 1 \).

   b) Is the linear operator \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 ; \mathbf{x} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} \) a self-adjoint operator on the inner product space \( (\mathbb{R}^2, \langle , \rangle) \)?

4. Let \( (V, \langle \cdot, \cdot \rangle) \) be a complex inner product space, \( T \in \mathbb{L}(V) \) a normal operator. Prove that \( T \) is self-adjoint if and only if all of the eigenvalues of \( T \) are real.

5. Let \( (V, \langle \cdot, \cdot \rangle) \) be a complex inner product space, \( T \in \mathbb{L}(V) \) a normal operator. Suppose that \( T^{10} = T^8 \). Prove that \( T \) is self-adjoint and that \( T^3 = T \).

6. Let \( (V, \langle \cdot, \cdot \rangle) \) be a complex inner product space, \( T \in \mathbb{L}(V) \) a normal operator. Prove or give a counterexample: if \( T^5 = 0 \in \mathbb{L}(V) \) then \( T = 0 \in \mathbb{L}(V) \).

7. Let \( (V, \langle \cdot, \cdot \rangle) \) be an inner product space (over \( F \)), \( T \in \mathbb{L}(V) \) a normal operator.
a) Let $F = \mathbb{C}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^4 = T$.

b) Let $F = \mathbb{R}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^4 = T$.

c) Let $F = \mathbb{R}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^5 = T$.

8. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space (over $\mathbb{C}$), $T \in L(V)$ an operator (not necessarily normal/self-adjoint!). Prove or give a counterexample:

a) if $T$ admits exactly two eigenvalues $1$ and $-i$ and $E(1, T) \subset E(-1, T)^\perp$ then $T$ is normal.

b) if $T$ admits exactly two eigenvalues $1$ and $-1$ and $E(1, T) = E(-1, T)^\perp$ then $T$ is self-adjoint.

9*. (Harder) Let $(V, \langle \cdot, \cdot \rangle)$ be a complex inner product space, $S, T \in L(V)$ normal operators. Prove: there exists a basis $B \subset V$ consisting of eigenvectors of both $S$ and $T$ if and only if $ST = TS$.

10*. (Harder) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with complex entries. Say that $A$ is normal if $AA^* = A^*A$, where $A^* = \overline{A}^t$ is the conjugate transpose. Give conditions on $a, b, c, d$ so that $A$ is normal and admits two distinct eigenvalues. What if you want $A$ to be normal and have exactly one eigenvalue?
Appendix 5

October 20 2014. DRP. Symmetric Group

Recall the symmetric group on $n$ letters $S_n$:

$$S_n = \text{Perm}\{1, \ldots, n\} = \{f : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid f \text{ bijective}\}.$$ 

Some facts about the symmetric group:

1. Any $\sigma \in S_n$ admit a decomposition into disjoint cycles $\sigma = C_1 \cdot \ldots \cdot C_k$, where $C_i = (i_1 \ldots i_{r_i})$, and $C_i \cap C_j = \emptyset$. The cycle type of $\sigma$ is the (nonincreasing) list $(r_1, \ldots, r_k)$, where $r_i$ is the length of the cycle $C_i$, $r_1 \geq r_2 \geq \ldots \geq r_k \geq 1$ and $\sum r_i = n$.

2. $\sigma, \tau \in S_n$ are conjugate if and only if they have the same cycle type.

3. The number of conjugacy classes is equal to $P(n)$, the number of integer partitions of $n$. Hence, the number of irreducible complex characters of $S_n$ is equal to $P(n)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n$ (we assume the sequence is always nonincreasing). The Young subgroup $Y_\lambda \subset S_n$ is the subgroup of $S_n$ consisting of those elements $\sigma \in S_n$ such that $\sigma(i) \in \{\lambda_j, \ldots, \lambda_j+1-1\}$, for every $i \in \{\lambda_j, \ldots, \lambda_j+1-1\}$.

**Exercise.** Let $\lambda = (3, 2)$ be a partition of 5. Show that $Y_\lambda$ is isomorphic to $S_3 \times S_2$, and deduce that $|Y_\lambda| = 3!2!$. More generally, show that, for any $\lambda$, a partition of $n$, $Y_\lambda$ is isomorphic to $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$; hence $|Y_\lambda| = \lambda_1! \cdots \lambda_k!$.

We are going to define representations of $S_n$ using Young subgroups: this will give us $P(n)$ representations and we will (eventually) show that we can use these representations to recover all irreducible representations of $S_n$.

Fix $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of $n$. Consider the set of left cosets $S_n/Y_\lambda$, and denote a set of coset representatives $\{x_1, \ldots, x_r\}$; thus

$$S_n/Y_\lambda = \{x_1 Y_\lambda, \ldots, x_r Y_\lambda\}.$$ 

**Exercise.** Show that $r = n!/\lambda_1! \cdots \lambda_k!$. For $n = 4$, $\lambda = (2, 2)$, find explicit representatives $x_1, \ldots, x_6 \in S_4$.

Let $M_\lambda = \mathbb{C}^r$ and define the following representation of $S_n$ on $M_\lambda$: first, fix a bijection between

$$\{x_1, \ldots, x_r\} \leftrightarrow \{e_1, \ldots, e_r\},$$

where $e_i$ are the standard basis vectors in $\mathbb{C}^r$. For $\sigma \in S_n$, define

$$\sigma \cdot e_i = e_j,$$

where $x_j Y_\lambda = (\sigma x_i) Y_\lambda$ and extend this linearly (ie, if $v = \sum a_i e_i$ then $\sigma \cdot v = \sum a_i \sigma \cdot e_i$).

For example, when $n = 3$, $\lambda = (2, 1)$, we have

$$Y_\lambda = \{\sigma \mid \sigma(1) = 1, \sigma(\{2, 3\}) = \{2, 3\}\} = \langle (23) \rangle.$$ 

Then, we can choose representatives $x_1 = e, x_2 = (12), x_3 = (13)$. Then, we would have

$$(123) \cdot e_1 = e_2,$$ 

since $(123)e Y_\lambda = (12) Y_\lambda$

$$(123) \cdot e_2 = e_3,$$ 

since $(123)(12) Y_\lambda = (13) Y_\lambda$

$$(123) \cdot e_3 = e_1,$$ 

since $(123)(13) Y_\lambda = e Y_\lambda$. 

1
**Exercise.** Determine the character $\chi_{(2,1)}$ of the representation $M_{(2,1)}$ (recall that characters are constant on conjugacy classes!). If you have time, try to determine the character $\chi_{(2,2)}$ of the representation $M_{(2,2)}$ above.

The representation we’ve just constructed is called the **permutation representation on** $S_n/Y_\lambda$.

Let’s now try and understand the character $\chi_\lambda$ of the representation $M_\lambda$ just constructed:

**Exercise.** Let $\sigma \in S_n$, $M_\lambda$ be the permutation representation constructed above (so you’ve fixed some representatives $\{x_1, \ldots, x_r\}$ etc). Suppose that $\sigma \in C_\mu$, where $C_\mu$ is a conjugacy class in $S_n$.

1. Explain why
   $$\chi_\lambda(\sigma) = |\{x_j \mid \sigma x_j Y_\lambda = x_j Y_\lambda\}|.$$  
   (Hint: the character is defined as the trace of any matrix representation of $\sigma$). Use this to deduce that
   $$\chi_\lambda(\sigma) = \frac{|\{\tau \in S_n \mid \tau^{-1}\sigma\tau \in Y_\lambda\}|}{|Y_\lambda|}.$$  
   (Hint: what happens if you choose different coset representatives?)

2. Suppose that $\tau \in S_n$ is such that $\tau^{-1}\sigma\tau \in Y_\lambda$. Let $\text{Cent}_{S_n}(\sigma)$ be the centraliser of $\sigma$. Show that $w = z\tau$ also satisfies $w^{-1}\sigma w \in H$, for any $z \in \text{Cent}_{S_n}(\sigma)$.

3. By considering the intersection $C_\mu \cap Y_\lambda$ and the previous exercise, show that
   $$|\{\tau \in S_n \mid \tau^{-1}\sigma\tau \in Y_\lambda\}| = |C_\mu \cap Y_\lambda||\text{Cent}_{S_n}(\sigma)|.$$  
   (Hint: can you find a bijection between the set on the LHS and $(C_\mu \cap Y_\lambda) \times \text{Cent}_{S_n}(\sigma)$)

4. Deduce that
   $$\chi_\lambda(\sigma) = \frac{n!}{\lambda_1! \cdots \lambda_k!} \frac{|C_\mu \cap Y_\lambda|}{|C_\mu|}.$$  

5. * Can you see how to count $|C_\mu \cap Y_\lambda|$? (This is quite difficult...! A formula is given on p.55 of Fulton-Harris; it’s the big ugly looking sum and product that appears just before ‘...where the sum is over all collections...’) it might help to look at some actual examples (ie, fix $n, \mu, \lambda$).
Math 110, Summer 2012: **Exam 1**

Instructor: George Melvin

Monday, 16th July 2012 - 10.15am-12pm

**Attempt at least THREE out of the following FIVE questions. You MAY ATTEMPT more than three questions: in this case, your best three answers will make up your overall score. Please CIRCLE BELOW THOSE QUESTIONS ATTEMPTED**

1. This is a closed book exam. Please put away all your notes, textbooks, calculators and portable electronic devices and turn your mobile phones to ‘silent’ mode.

2. Explain your answers **CLEARLY** and **NEATLY**. State all theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and neatly.

3. Correct answers without appropriate justification will be treated with skepticism.

4. Write your name on this exam and any extra sheets you hand in.

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**Question 1:** /25  
**Question 2:** /25  
**Question 3:** /25  
**Question 4:** /25  
**Question 5:** /25  
**Total:** /75

Name: ________________________________

SID: ________________________________
1. Let $V$ be a $K$-vector space, for some number field $K$. Let $E \subset V$ be a nonempty subset of $V$.

i) (4 pts) Define what it means for $E$ to be linearly independent (over $K$). Define what it means for $E$ to be linearly dependent (over $K$).

ii) (3 pts) Suppose that $E$ is linearly independent and let $F \subset E$ be a nonempty subset. Prove that $F$ is linearly independent.

iii) (5 pts) Suppose that $E$ is linearly dependent. Prove that there exists $v \in E$ such that $v$ can be written as a linear combination

$$v = c_1 v_1 + \ldots + c_k v_k, \quad \text{with } c_i \in K, v_j \in E.$$  

iv) (7 pts) Suppose that $E = \{e_1, \ldots, e_n\}$ is linearly independent. Let $w \in V$ be such that $w \notin \text{span}_K E$. Prove that $E \cup \{w\}$ is linearly independent.

v) (6 pts) Show that

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\} \subset \text{Mat}_2(\mathbb{Q}),$$

is linearly independent and extend $E$ to a basis of $\text{Mat}_2(\mathbb{Q})$. 
2. i) (3 pts) Let $B = (b_1, \ldots, b_n) \subset V$ be an ordered subset of the $K$-vector space $V$. Define what it means for $B$ to be an ordered basis of $V$ (You can use ANY definition here.)

ii) (2 pts) Suppose that $E \subset V$ is a linearly independent subset of a finite dimensional $K$-vector space $V$. What is the allowed possible size of $E$?

iii) (6 pts) Suppose that $V$ is a $K$-vector space such that $\dim_K V = n$. Let $E \subset V$ be a linearly independent subset of size $|E| = n$. Prove that $\text{span}_K E = V$. (Hint: Use a ‘proof by contradiction’ argument.)

iv) (6 pts) Consider the ordered subset $B = (f_1, f_2, f_3) \subset \mathbb{Q}^{\{1,2,3\}} = \{ f : \{1, 2, 3\} \to \mathbb{Q} \}$, where

$$f_1(1) = 1, \quad f_1(2) = 0, \quad f_1(3) = -1, \quad f_2(1) = 1, \quad f_2(2) = 0, \quad f_2(3) = 1, \quad f_3(1) = 0, \quad f_3(2) = 1, \quad f_3(3) = 1.$$

Prove that $B$ is linearly independent. Deduce that $B$ is a basis of $\mathbb{Q}^{\{1,2,3\}}$.

v) (5 pts) Let $S = (e_1, e_2, e_3) \subset \mathbb{Q}^{\{1,2,3\}}$ be the standard ordered basis of $\mathbb{Q}^{\{1,2,3\}}$. Determine the change of coordinate matrix $P_{B \leftarrow S}$.

vi) (3 pts) Suppose that $f \in \mathbb{Q}^{\{1,2,3\}}$ is such that

$$[f]_B = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Is $f \in \text{span}_{\mathbb{Q}} \{ f_1, f_3 \}$? Justify your answer.
3. i) (6 pts) Define the image $\text{im} f$ of a linear morphism $f : V \to W$ and the rank of $f$, $\text{rank} f$. Define the rank of an $m \times n$ matrix $A \in \text{Mat}_{m,n}(\mathbb{K})$, $\text{rank} A$.

ii) (7 pts) Prove: if $\text{rank} f = \dim V$ then $f$ is surjective.

iii) (5 pts) Prove: if $A \in \text{Mat}_{m,n}(\mathbb{K}), B \in \text{Mat}_{n,p}(\mathbb{K})$ and $\text{rank} A = r, \text{rank} B = s$, then $\text{rank} AB \leq r$.

iv) (7 pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$ 

Determine $r = \text{rank} A$ and find $P, Q \in \text{GL}_3(\mathbb{Q})$ such that

$$Q^{-1}AP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$
4. i) (4 pts) Let \( f \in \text{End}_\mathbb{C}(V) \), with \( V \) a finite dimensional \( \mathbb{C} \)-vector space. Define what it means for \( \lambda \in \mathbb{C} \) to be an eigenvalue of \( f \). Define the geometric and algebraic multiplicity of \( \lambda \).

ii) (4 pts) Let \( f \in \text{End}_\mathbb{C}(V) \), with \( V \) a finite dimensional \( \mathbb{C} \)-vector space. Define what it means for \( f \) to be diagonalisable. Give a criterion for \( f \) to be diagonalisable using the notions of geometric and algebraic multiplicity of eigenvalues.

iii) (7 pts) Let \( f \in \text{End}_\mathbb{C}(V) \), where \( \dim V = 7 \). Suppose that \( f \) is non-surjective, diagonalisable and such that \( \dim \text{im} f = 1 \). Prove that \( f \) admits exactly one nonzero eigenvalue \( \lambda \) and that \( E_\lambda = \text{im} f \), where \( E_\lambda \) is the \( \lambda \)-eigenspace.

Consider the endomorphism
\[
f : \text{Mat}_2(\mathbb{C}) \to \text{Mat}_2(\mathbb{C}) ; \ A \mapsto A + A^t,
\]
where \( A^t \) is the transpose of \( A \).

iv) (4 pts) Determine the eigenvalues of \( f \) and their algebraic multiplicities.

v) (6 pts) Prove that \( f \) is diagonalisable and find a basis \( B \subset \text{Mat}_2(\mathbb{C}) \) such that \([f]_B\) is diagonal.

For iv)-v) you may want to use the standard ordered basis
\[
S = (e_{11}, e_{12}, e_{21}, e_{22}) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \subset \text{Mat}_2(\mathbb{C}).
\]
5. Consider the following endomorphism

\[ L_A : \text{Mat}_n(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C}) ; \ B \mapsto AB, \quad \text{where} \ A \in \text{Mat}_n(\mathbb{C}). \]

i) (4 pts) Define what it means for \( L_A \) to be nilpotent. Define what it means for \( A \) to be nilpotent.

ii) (2 pts) Define the exponent of \( L_A \), \( \eta(L_A) \). Define the exponent of \( A \), \( \eta(A) \).

iii) (4 pts) Prove: \( A \) is nilpotent if and only if \( L_A \) is nilpotent.

Now suppose that \( n = 2 \) and \( A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \).

iv) (2 pts) Using iii) deduce that \( L_A \) is nilpotent. What is \( \eta(L_A) \)?

v) (3 pts) Let \( S = (e_{11}, e_{12}, e_{21}, e_{22}) \) be the standard ordered basis of \( \text{Mat}_2(\mathbb{C}) \). Determine \( X = [L_A]_S \).

vi) (7 pts) Determine an ordered basis \( B = (b_1, b_2, b_3, b_4) \subset \text{Mat}_2(\mathbb{C}) \) such that \( [L_A]_B \) is a block diagonal matrix, each block being a 0-Jordan block.

vii) (3 pts) Determine the partition associated to \( X \), \( \pi(X) \). Is \( X \) similar to the following matrix

\[ Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]?

Justify your answer.
Appendix 7

University of California
Berkeley
2013
The Graduate Division
and
The Graduate Council
Advisory Committee for GSI Affairs
hereby recognize
George Melvin
as an
Outstanding Graduate Student Instructor

Andrew J. Szeri
Dean
Graduate Division

Rosemary A. Joyce
Associate Dean
Graduate Division

Laura Stoker
Chair
Advisory Committee for GSI Affairs

Linda von Hoene
Director
GSI Teaching and Resource Center
advanced linear algebra notes

Debra Lewis <lewis@ucsc.edu> 12 February 2014 at 06:19
To: gmelvin@berkeley.edu

George,

I'm teaching the advanced linear algebra course at UCSC this quarter, and one of the students, Sergey Kojoian, highly recommended your lecture notes. I'm on UCSC's Committee on Computing and Telecommunications, and on Monday we were discussing the propagation across the web of course materials that weren't intended for mass distribution, so... would you be willing to share your notes with my class, and if so, how? I use Piazza, and could post the pdf there, or put a link to your website. If you'd prefer not to have them distributed to Slugs, quite alright.

Thanks.

Deb