MARCH 2 LECTURE

TEXTBOOK REFERENCE:
- Vector Calculus, Colley, 4th Edition: §3.3

VECTOR FIELDS AND FLOW LINES

LEARNING OBJECTIVES:
- Gain familiarity with plotting vector fields.
- Understand the concept of a flow line.
- Learn how to determine flow lines in simple cases.

KEYWORDS: vector fields, flow lines

In this lecture we consider more general vector-valued functions known as vector fields. Vector fields in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) have natural interpretations in terms of fluid flow, and the mathematical notion of a flow line embodies this idea.

Vector fields

In the last lecture we introduced paths as continuous functions \( \mathbf{x} : I \rightarrow \mathbb{R}^n \), where \( I \subset \mathbb{R} \) is an interval of real numbers. The image of a path is realised as a curve:

\[
\begin{array}{c}
\downarrow \\
t \\
\mathbf{x}(t)
\end{array}
\]

When \( \mathbf{x} \) is differentiable (so that the component functions of \( \mathbf{x}(t) \) are all differentiable), we can form the velocity vector \( \mathbf{x}'(t_0) \) at \( t = t_0 \). In this way, we obtain a collection of vectors based at points along the curve described by the path \( \mathbf{x} \).

Keep this example in mind as we go through today’s lecture.

A vector field on \( \mathbb{R}^n \) is a (continuous\(^1\)) function

\[
\mathbf{F} : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \\
(x_1, \ldots, x_n) \mapsto \begin{bmatrix} F_1(x_1, \ldots, x_n) \\ \vdots \\ F_n(x_1, \ldots, x_n) \end{bmatrix}
\]

which assigns to each point in \( X \) a vector in \( \mathbb{R}^n \).

When \( n = 2, 3 \), we will represent a vector field \( \mathbf{F} \) visually as a collection of vectors based at points in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

Example:

\(^1\)We will discuss what continuity means for functions of several variables in a couple of weeks.
1. Consider the vector field \( \mathbf{F}(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \), defined for every \((x, y) \in \mathbb{R}^2\). We can represent this vector field.

General features:

- \(|\mathbf{F}(x, y)| \) grows as \((x, y)\) grows.
- Vectors in quadrant 1, 3 have >0 slope.
- Vectors in quadrant 2, 4 have <0 slope.

2. Consider the vector field \( \mathbf{F}(x, y) = \begin{bmatrix} 1 \\ (x - y) \end{bmatrix} \), defined for every \((x, y) \in \mathbb{R}^2\). We can represent this vector field.
General features:

- Constant along lines $y = x + a$
- Vectors always point to the right

Observe that along the line $y = x$ the vector field gives horizontal vectors pointing to the right: when $x = y$, the vector field outputs the vector $F(x, x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so that, for any point $(x, x)$ on the line, we draw the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3. Consider the vector field $F(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$, defined for every $(x, y) \in \mathbb{R}^2$. We can represent this vector field

![Vector Field Diagram]

$\begin{cases} m < 0 \\ m > 0 \end{cases}$

General features:

- $|F(x, y)|$ grows as $|x + y|$ grows.
- $F(x, y) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow F(x, y) \perp \begin{bmatrix} x \\ y \end{bmatrix}$

Observe that $F(x, y) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$, for every $(x, y)$.

Remark: given a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$, it can be difficult to represent the vector field visually. However, it's important to think about the general features of the vector field: Do the vectors point in a general direction in certain regions of the plane/space? Can you identify where the vectors are zero? Can you identify where the vectors have large/small magnitude? Can you identify where the vectors are horizontal/vertical?
CHECK YOUR UNDERSTANDING
Sketch the given vector fields in $\mathbb{R}^2$.

1. $F(x, y) = \begin{bmatrix} 2 \\ x \end{bmatrix}$

   **Flow lines:**
   
   $x(t) = \begin{bmatrix} 2t \\ e^{t^2 + a} \end{bmatrix}$, $a$ constant

   **Check:**
   
   $x'(t) = \begin{bmatrix} 2 \\ 2te^t \end{bmatrix}$

   $F(x(t)) = E \left( \begin{bmatrix} 2t \\ e^{t^2 + a} \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2te^t \end{bmatrix}$

   $x(t) = \begin{bmatrix} e^{-t^2} \\ e^{-t^2 + a} \end{bmatrix}$, $a$ constant

   **Check:**
   
   $x'(t) = \begin{bmatrix} e^{-t^2} \\ -2te^{-t^2} \end{bmatrix}$

   $F(x(t)) = E \left( \begin{bmatrix} e^{t^2} \\ e^{-t^2 + a} \end{bmatrix} \right) = \begin{bmatrix} e^{-t^2} \\ -2te^{-t^2} \end{bmatrix}$

   3. $F(x, y) = \begin{bmatrix} y^2 \\ y \end{bmatrix}$

   **Flow lines:**
   
   $x(t) = \begin{bmatrix} \frac{1}{2} e^{t^2} + a \\ \frac{1}{2} e^{t^2} \end{bmatrix}$, constant

   $x'(t) = \begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$

   $F(x(t)) = E \left( \begin{bmatrix} \frac{1}{2} e^{2t} + a \\ e^{t^2} \end{bmatrix} \right) = \begin{bmatrix} e^{2t} \\ -t \end{bmatrix}$

   Indpt $x$, $y$ => constant along lines $x = c$, $y = c$
Vector fields in nature:

Vector fields arise in lots of places. You've already been looking at vector fields since you were a child.

Figure 1: Wind reports are represented as vector fields

Other examples:

Figure 2: Vector field describing relationship between buy/sell strategy for two stocks in a certain market

Figure 3: Vector field describing relationship between two stocks with flow lines
Definition: Let $F$ be vector field in $\mathbb{R}^n$. A flow line of $F$ is a differentiable path $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\gamma'(t) = F(\gamma(t)), \quad \text{for every } t \in I.$$ 

A flow line is a path $\gamma$ whose velocity vector at time $t$ is the value of the vector field at the point $\gamma(t)$ on the image curve of $\gamma$.

Example:

1. Consider the vector field $F(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$. The vector field is plotted above.

The curve $\gamma(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ is a flow line for $F$: indeed,

$$\gamma'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

and

$$F(\gamma(t)) = F\left(\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}\right) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} = \gamma'(t), \quad \text{for all } t$$

Think about the relationship between the path $\gamma(t)$ and the general ‘shape’ of $F$.

2. Consider the vector field $F(x, y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The vector field is plotted below.

A flow line $\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ must satisfy

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \gamma'(t) = F\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

That is, we require

$$x'(t) = 2, \quad y'(t) = 3 \quad \implies \quad x(t) = 2t + a, \quad y(t) = 3t + b$$

Hence, the flow line must take the form

$$\gamma(t) = \begin{bmatrix} 2t + a \\ 3t + b \end{bmatrix}$$

This is a line in the direction $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ passing through the point $(a, b)$. 