March 14 Lecture

Textbook Reference:
- Vector Calculus, Colley, 4th Edition: §2.2

Limits & Continuity

Learning Objectives:
- Understand the concept of limit for a function of several variables.
- Learn how to determine limits for simple functions.

Keywords: limit

Today we introduce the notion of a limit for a function of several variables. We will introduce the intuitive notion of a limit and see how to determine the limit of some rational functions. We will define what it means for a function of several variables to be continuous.

Limits of functions of several variables

Let

\[ f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m, \ x \mapsto f(x) = (f_1(x), \ldots, f_m(x)) \]

be a function of several variables.

Intuitive notion of limit I

Intuitively, the limit of f as \( x \) tends to \( a \) is the vector \( L \in \mathbb{R}^m \) that \( f(x) \) approaches whenever \( x \) is near to \( a \) (but not equal to \( a \)), should such a vector \( L \) exist.

In the case that \( L \) exists, we write

\[ \lim_{x \to a} f(x) = L \]

We need to be more precise with what we mean by approaches and near to.
Intuitive notion of limit II

Intuitively,

\[ \lim_{x \to a} f(x) = L \]

means that we can make \(|f(x) - L|\) arbitrarily small (i.e. as close to 0 as we please) by keeping \(|x - a|\) sufficiently small (but nonzero).

Example: Consider the function

\[ f : \mathbb{R}^2 \to \mathbb{R}^3, \quad (x, y) \mapsto (x, y, 2y) \]

Intuitively, as \(x = (x, y)\) gets close, but not equal, to \(a = (1, 1)\) we expect that \(f(x)\) gets close to \(L = (1, 1, 2)\): for \(x\) such that

\[ \left| x - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right| = \sqrt{(x-1)^2 + (y-1)^2} \]

is sufficiently small (i.e. \(x\) is sufficiently close to \((1, 1)\)), we find that we can make

\[ \left| f(x) - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right| = \sqrt{(x-1)^2 + (y-1)^2 + (2y-2)^2} = \sqrt{(x-1)^2 + 5(y-1)^2} \]

arbitrarily small.

For example, to make \(|f(x) - L| < 0.01\) we can take those \(x \in \mathbb{R}^2\) such that \(|x - a| < \frac{1}{1000} = 0.001\): indeed, in this case

\[ |f(x) - a| = \sqrt{(x-1)^2 + 5(y-1)^2} \leq |x - 1| + 5|y - 1| \leq 5|x - 1| + 5|y - 1| \leq \frac{5}{1000} + \frac{5}{1000} = \frac{1}{100} = 0.01 \]

Notes:
Check your understanding

1. Determine $\delta > 0$ such that $|f(x) - L| < 1/500$ whenever $|x - a| < \delta$.

2. Let $\epsilon > 0$. Determine $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

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**Rigorous definition of limit**

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. We write $\lim_{x \to a} f(x) = L$ if, given any $\epsilon > 0$, you can find $\delta > 0$ such that

$$
\text{if } x \in X \text{ and } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon
$$

We call $L$ the limit of $f$ as $x$ tends to $a$.

**Remark:** You should have seen a similar definition for the limit of a single variable function in Calculus I.

**Determining limits of several variable functions**

In general, verifying the $\epsilon - \delta$ condition above gets very messy very quickly. One important observation is the following:

**Observation:** Let $a \in \mathbb{R}^n$, where $n = 2, 3$. If $x \in \mathbb{R}^n$ satisfies $|x - a| < \delta$ then $x$ lies inside the disc/sphere of radius $\delta$, centred at $a$.

Therefore, the statement $\lim_{x \to a} f(x) = L$ means that, as $x$ moves towards $a$, $f(x)$ moves towards $L$, irrespective of the path $x$ takes to get close to $a$. 
Example: Consider the function

$$f : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}, \ (x, y) \mapsto \frac{2x^2 + y^2}{x^2 + y^2}$$

This function is not defined at \((0, 0)\) - we can’t evaluate the quantity \(\frac{0}{0}\). However, we could still ask whether \(\lim_{x \to (0, 0)} f(x)\) exists.

If this limit did exist then it will be the same no matter how we approach \((0, 0)\). For example, if we approach \((0, 0)\) along the \(x\)-axis, where \(y = 0\), then

$$f(x, 0) = \frac{2x^2 + 0}{x^2 + 0} = 2, \quad x \neq 0$$

This means that the function \(f\) is constant along the \(x\)-axis. Similarly, if we approach \(0\) along the \(y\)-axis, where \(x = 0\), then

$$f(0, y) = \frac{0 + y^2}{0 + y^2} = 1 \quad y \neq 0$$

We see that approaching \((0, 0)\) from two different directions gives rise to two distinct values for the limit. Therefore, \(\lim_{x \to (0, 0)} f(x)\) does not exist.