In this lecture we will describe some new coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Coordinates in the plane**

Consider the plane $\mathbb{R}^2$ - this is a flat two-dimensional surface that is infinite in all directions. The basic question is

**Question:** how can we describe points in the plane?

To the Greeks a point just was: we would care about describing points when they appeared in a problem of geometry and were a (un)known distance from another point.

Many centuries later, Descartes (and, independently, Fermat) came up with the following revolutionary idea: fix a point in the plane (call it $O$), choose two perpendicular fundamental directions (let’s call them $\mathbf{i}$ and $\mathbf{j}$) and basic units of length, describe points relative to these fundamental directions. This, of course, leads to our usual Cartesian (or rectangular) description of the plane using $(x, y)$ coordinates.

![Coordinate system diagram](diagram.png)

In linear algebra terms, the vectors $\mathbf{i}, \mathbf{j}$ are linearly independent and therefore provide a basis of $\mathbb{R}^2$. We could extend this idea by choosing any two linearly independent vectors $\mathbf{u}, \mathbf{v}$ to determine a coordinate system on $\mathbb{R}^2$: 
Polar coordinates: a useful coordinate system in the plane, called the polar coordinate system, is defined as follows: fix an origin $O$. Any point $P$ (distinct from the origin $O$) lies on a unique circle of some radius $r$. To determine precisely where the point $P$ is on the circle, we fix a line through the origin (which we assume is horizontal) and measure (counterclockwise) the angle $\theta$ subtended by $P$ from this line.

The point $P$ is represented by the pair $(r, \theta)$, the polar coordinates of $P$. To remove ambiguity, always choose $0 \leq \theta < 2\pi$.

Convention: Sometimes we will also allow $r$ to take negative values, to be interpreted as follows: given polar coordinates $(r, \theta)$, with $r < 0$, consider the ray making angle $\theta$ with the $x$-axis, and instead of moving $|r|$ units away from the origin along this ray, go $|r|$ units in the opposite direction.

Remark: Restricting $0 \leq \theta < 2\pi$, $r \geq 0$, ensures that any point in the plane, apart from the origin $O$, has a unique set of polar coordinates.

Example:
1. The point \( P = (2, 2) \) (in Cartesian coordinates) lies on a circle of radius \( \sqrt{2^2 + 2^2} = 2\sqrt{2} \), and we have \( \tan \theta = 1 \). Hence, since \( x, y > 0 \), we must have \( \theta = \frac{\pi}{4} \). Therefore, in polar coordinates the point \( P \) is represented by \((r, \theta) = (2\sqrt{2}, \theta)\).

2. Consider the point \( P \) which is represented by \((5, \pi/6)\) in polar coordinates. Then, \( P \) lies in the first quadrant on the arc of the circle, centred at \( O \), of radius 5. Recalling some basic trigonometry we have, in Cartesian coordinates, \( P = (x, y) \), where \( x = r \cos \theta, \ y = r \sin \theta \) i.e. \( P = (5\sqrt{3}/2, 5/2) \).

3. The origin is weird: it is given, in polar coordinates, by \((0, \theta)\), for any \( \theta \).

Since a point \( P \) in the plane doesn’t care about how we represent it (it just is, as the Greeks would say), we should be able to change between polar and Cartesian coordinate representations for \( P \) (analogous to change-of-coordinate transformations in linear algebra).

**Cartesian ↔ polar coordinate transformation**

<table>
<thead>
<tr>
<th>Polar to Cartesian:</th>
<th>( x = r \cos \theta ) (1)</th>
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<tbody>
<tr>
<td>( y = r \sin \theta )</td>
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<table>
<thead>
<tr>
<th>Cartesian to polar:</th>
<th>( r^2 = x^2 + y^2 ) (2)</th>
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<tbody>
<tr>
<td>( \tan \theta = \frac{y}{x} )</td>
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**Caution:** the Cartesian to polar change-of-coordinate formula in (2) do not specify \((r, \theta)\) uniquely in terms of \( x, y \). Read through p.64 of the textbook for discussion.

**Polar equations:** we can describe geometric objects in the plane using equations in polar coordinates.

1. In polar coordinates \((r, \theta)\), a circle having radius \( c \), centred at the origin, is defined by the equation \( r = c \).

2. The straight line through the origin with slope \( m \) is given by the equation \( \tan \theta = m \).

3. The vertical line through \((0, 1)\) is given by \( r = \sec \theta \): rearranging this equation gives \( 1 = r \cos \theta = x \).

4. The equation \( r = 2 \cos \theta \) describes a circle of radius 1 centred at \((1, 0)\): multiplying both sides by \( r \) gives

\[
r^2 = 2r \cos \theta \quad \Rightarrow \quad x^2 + y^2 = 2x.
\]

Completing the square gives

\[
(x - 1)^2 + y^2 = 1.
\]
Remark: determining the shapes described by a polar equation is tricky and takes some getting used to. Can you see what shape is described by the polar equation \( r = \theta \)?

**Coordinates in space**

The Cartesian coordinates in the plane can be extended to space: we add in a new \( z \) coordinate, where \( z \) measures units distance in the direction \( \mathbf{k} \overset{\text{def}}{=} \mathbf{i} \times \mathbf{j} \). As in the \( \mathbb{R}^2 \) case, we could also describe points in space (once we’ve fixed an origin \( O \)) by giving three linearly independent vectors \( u, v, w \) and determining a new coordinate system with respect to the resulting basis.

**Cylindrical coordinates**: Polar coordinates provide us with a coordinate system in the plane and we can extend this to a coordinate system in \( \mathbb{R}^3 \).

Given a point \( P \) in space, use polar coordinates to describe the projection of \( P \) onto the \( xy \)-plane: denote this projection \((r, \theta)\). Then, \( P \) can be described by the triple \((r, \theta, z)\). We say that \((r, \theta, z)\) obtained in this way are the **cylindrical coordinates** of \( P \).

The terminology is justified by considering the following diagram:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \quad (1) \]
\[ z = z \]
\[ r^2 = x^2 + y^2 \]
\[ \tan \theta = \frac{y}{x} \quad (2) \]

\( z = z \)
Remark:

1. As with polar coordinates, all points in $\mathbb{R}^3$ except for the $z$-axis have a unique set of cylindrical coordinates. Any point $(0, 0, c)$ on the $z$-axis has cylindrical coordinates $(0, \theta, c)$, where $\theta$ can be any angle.

2. Cylindrical coordinates are useful when studying objects possessing rotational symmetry (about the $z$-axis).

Example:

1. The surface in $\mathbb{R}^3$ described by $r = c$ is the cylinder, centred at the origin, parallel to the $z$-axis, and having radius $c$. In Cartesian coordinates, we see that a cylinder (parallel to the $z$-axis) is therefore given by the equation

$$\sqrt{x^2 + y^2} = c \quad \text{or, equivalently} \quad x^2 + y^2 = c^2.$$ 

This example highlights an important point: \textit{if an equation does not contain a coordinate, then the resulting object described by the equation extends infinitely in both directions parallel to the axis of the missing coordinate.}

2. The surface in $\mathbb{R}^3$ described by the equation $\tan \theta = m$, is the plane containing the $z$-axis and the line $y = mx$.

3. The surface in $\mathbb{R}^3$ described by the equation $z^2 + r^2 = 400$, $r \in \mathbb{R}$, is a sphere of radius 20 centred at the origin: in Cartesian coordinates the equation becomes

$$z^2 + r^2 = 400 \quad \implies \quad z^2 + x^2 + y^2 = 20^2$$

If $(x, y, z)$ lies on the surface described by this equation then it must be at distance 20 from the origin. All points in space at a fixed distance from the origin define a sphere centred at the origin.