FEBRUARY 12 LECTURE

In this exercise we will have a geometric stretch and begin to flex our mathematical muscles. For our workout we will investigate the determination of the foci of an ellipse by a method known as Dandelin’s spheres (discovered in 1822 by Germinal Pierre Dandelin, a Belgian mathematician).

Warm-up exercises

1. [Some geometry] Consider the following geometric figure $C$, known as a cone (for obvious reasons!).

(a) Imagine you dropped a ball $S$ (mathematicians often call balls, ‘spheres’) of radius $r$ into the cone. Describe the set (= collection) of points $\mathcal{P}$ on the cone that will be touching the ball $S$.

(b) Drop a larger ball $S'$ of radius $s > r$ into the cone. Describe the set of points $\mathcal{P}'$ on the cone that will be touching the ball $S'$.

(c) Can you state a geometric relationship between the sets $\mathcal{P}$ and $\mathcal{P}'$?
2. [Some linear algebra] Recall the **dot product**: let \( \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n \).

Then, the dot product is the real number

\[
\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n
\]

(a) **Fill in the blanks!** Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \) be vectors.

i. \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \) ________________

ii. \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \) ________________

iii. If \( \mathbf{u} \cdot \mathbf{v} = 0 \) then

\[
(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \) ________________

What well-known Theorem is this?

(b) Suppose \( n = 3 \). Give a geometric interpretation of the quantity

\[
|\mathbf{u}| \overset{\text{def}}{=} \sqrt{\mathbf{u} \cdot \mathbf{u}}
\]

(c) Imagine a ball sitting in front of you. Choose a point \( P \) outside the ball.
Imagine drawing a straight line \( L \) from \( P \) to a point \( Q \) on the ball so that the line \( L \) is tangent to the ball. Choose another point \( Q' \) on the ball \((Q \neq Q')\) given by drawing another line \( L' \) starting at \( P \) that is tangent to the ball. (It may be useful to draw a picture below!)

What is the relationship between the length of \( L \) and the length of \( L' \)?