Instructions:
• You must attempt Problem 1.
• Please attempt at least five of Problems 2, 3, 4, 5, 6, 7, 8.
• If you attempt all eight problems then your final score will be the sum of your score for Problem 1 and the scores for the five remaining problems receiving the highest number points.
• Calculators are not permitted.

1. (20 points) True/False:
   (a) Let $F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$. If $F$ is conservative then $u_y = v_x$.
   (b) Every function $f(x,y)$ defined on $\mathbb{R}^2$ has a local maximum or local minimum.
   (c) If $f(x,y)$ is a continuous function then $\int_1^0 \int_1^x f(x,y) dy dx = \int_1^0 \int_y^1 f(x,y) dx dy$.
   (d) If $C$ is a closed oriented curve and $F$ is a vector field satisfying $\int_C F \cdot dr = 0$ then $F$ is conservative.
   (e) If $u, v$ are vectors then $\frac{\partial u \times v}{\partial t}$ is the area of the parallelogram spanned by $u, v$.
   (f) $\nabla f = \frac{d}{dt} f(x + t, y + t, z + t)$.
   (g) The vector line integral of $F$ along the ellipse $x^2 + 5y^2 = 1$ is zero.
   (h) If $u, v, w \in \mathbb{R}^3$ lie in a common plane then $u \cdot (v + w) \times w = 0$.
   (i) Consider the surface $S : z^2 = f(x,y)$. If $P = (x,y, \sqrt{f(x,y)})$ is a point on $S$ with maximal distance from $(0,0,0)$ then $P$ is a local maximum of $g(x,y) = x^2 + y^2 + f(x,y)$.
   (j) Using linear approximation, the value $\sqrt{101 \cdot 10002}$ is estimated as $1000 + 5 + \frac{1}{10}$.

2. Let $P = (2,1,1)$, $Q = (1,0,-1)$, $R = (0,-1,2)$.
   (a) Compute $\overrightarrow{PQ} \times \overrightarrow{PR}$.
   (b) Write down the equation of the plane $\Pi : ax + by + cz = d$ containing the points $P, Q, R$.
   (c) Find the distance from the origin to $\Pi$.

3. Let $F(x,y) = \begin{bmatrix} x^2 - 2xye^{-x^2} + 2y \\ e^{-x^2} + 2x + \cos(y) \end{bmatrix}$.
   (a) Show that $F$ is conservative by finding a potential function $f(x,y)$ such that $\nabla f = F$.
   (b) If $C$ is the oriented curve going from $(1,0)$ to $(-1,0)$ along the semicircle $x^2 + y^2 = 1$, $y \geq 0$, evaluate $\int_C F \cdot dr$.

4. Consider the spheres
   $S_1 : x^2 + y^2 + z^2 = 6$, \quad $S_2 : (x-3)^2 + y^2 + (z+1)^2 = 16$. 
(a) Determine the tangent plane to $S_1$ at the point $(1,1,2)$ and the tangent plane to $S_2$ at the point $(-1,0,-1)$.  
(b) Find a parameterisation of the line of intersection $L$ of the tangent planes.  
(c) Determine the distance from the centre of $S_2$ to $L$. 

5. (a) Classify the critical points of the function  
$$f(x,y) = x^3 - y^2 - xy + 1$$  
(b) Determine the absolute maximum of $f(x,y)$ on the triangle having vertices $(0,0), (-1,0), (0,1)$ 
*Hint: consider the extrema of $f(x,y)$ on the interior of the triangle and on the boundary of the triangle.*

6. (a) Let $f(x,y) = x + y$ and $D$ be the region bounded between the $x$-axis and and the parabola $y = 1 - x^2$, $1 \leq x \leq 5$. Compute  
$$\int \int_D f \, dA$$  
(b) Evaluate the integral by changing the order of integration  
$$\int_0^3 \int_{x^2}^9 xe^{-y^2} \, dy \, dx$$

7. (a) Given below is the level curve diagram of a function $f(x,y)$. Mark the points on the circle $C$ where the extrema to the constrained optimisation problem  
$$\text{max. } f(x,y)$$  
$$\text{subject to } C$$  
can occur. 

(b) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Find the point $P = (x_0, y_0)$ on the ellipse $E : x^2 + 4y^2 = 1$ such that $\overrightarrow{OP} \cdot \mathbf{v}$ is maximised.
8. (a) Draw the region $D'$, described in polar coordinates by $\pi/4 \leq \theta \leq 3\pi/4$, $0 \leq r \leq 2$.

(b) Let $f(x, y) = y - x$. Using the linear change of coordinate formula, compute

$$\int \int_{D'} f dA$$

(Hint: if $\theta \in [0, 2\pi]$ then $M_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is the matrix corresponding to the ‘rotate by $\theta$ counterclockwise’ linear transformation.)