

APRIL 4 LECTURE

Textbook Reference:
- Vector Calculus, Colley, 4th Edition: §2.6

Directional Derivative

Learning Objectives:
- Learn how to compute the directional derivative.
- Learn how to compute the tangent line/plane to a level curve/surface.

Directional Derivatives: Given a differentiable function \( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, \ a \in X, \ v \in \mathbb{R}^n \), the directional derivative of \( f \) at \( a \) in the direction \( v \) is

\[
D_v f(a) = \nabla f(a) \cdot v
\]

1. Compute the directional derivative of \( f(x, y) = x^2 + 3xy + y^2 \) at the point \((2, 1)\) in the direction that points towards the origin.

2. Find a nonzero vector \( v \in \mathbb{R}^2 \) so that \( D_v f(2, 1) = 0 \)

3. Let \( w \in \mathbb{R}^2 \) be a vector satisfying \( w \cdot v > 0 \) and \( w \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} < 0 \). Is \( f \) increasing in the direction \( w \) at \((2, 1)\)? (Recall that, if \( \mathbf{a} \cdot \mathbf{b} > 0 \) (resp. \( < 0 \)) then angle between \( \mathbf{a} \), \( \mathbf{b} \) is acute (resp. obtuse))
Tangent lines/planes of level sets: Let $S$ be a level set of $f : X \subset \mathbb{R}^n \to \mathbb{R}$, where $n = 2, 3$. This means $S = \{x \mid f(x) = c\}$, for some $c$. Then, $\nabla f(a)$ is

- perpendicular to the tangent line to $S$ at $a$ ($n = 2$),
- normal to the tangent plane to $S$ at $a$ ($n = 3$).

1. Consider the curve $y^2 = x^3 - 4x + 1$.

(a) Find a function $h(x, y)$ so that the curve is a level curve of $h$

(b) Compute the tangent line to the curve at $(2,1)$.

(c) Determine the point $(x, y), y > 0$, where the tangent line is horizontal.
2. Consider the paraboloid $z = x^2 + y^2$. This is the graph of the function $f(x, y) = x^2 + y^2$.

(a) Find a function $g(x, y, z)$ so that the paraboloid is a level set of $g$.

(b) Determine the tangent plane to the paraboloid at $(1, 2, 5)$

(c) Suppose that $z = f(x, y)$ is a surface. Generalise your approach above to determine the tangent plane to the surface at $(a, b, f(a, b))$. 