April 16 Lecture

Textbook Reference:
- Vector Calculus, Colley, 4th Edition: §6.1

Reparameterisation

Learning Objectives:
- Understand the notion of reparameterisation of a path.
- Understand the effect of reparameterisation on vector line integrals.

Keywords: reparameterisation, vector line integrals along curves

Today we will investigate the effect that reparameterisation has on vector line integrals.

Reparameterisations

Consider the $C^1$-path

$$
\mathbf{x}(t) = \begin{bmatrix} t \\ 2t + 1 \end{bmatrix}, \quad t \in [0, 2]
$$

whose image curve is the straight line segment between (0, 1) and (2, 5).

The same line segment may also be parameterised by the path

$$
\mathbf{y}(t) = \begin{bmatrix} 2t \\ 4t + 1 \end{bmatrix}, \quad t \in [0, 1]
$$

Remark: It’s important to remember that $\mathbf{x}$ and $\mathbf{y}$ are different paths describing the same curve (i.e. the line segment).

The paths $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are, of course, related:

$$
\mathbf{x}(2t) = \mathbf{y}(t), \quad \mathbf{x}(t) = \mathbf{y}(t/2)
$$

We say that $\mathbf{y}$ is a reparameterisation of $\mathbf{x}$ (and $\mathbf{x}$ is reparameterisation of $\mathbf{y}$).

More generally:
Reparameterisation of paths

Let \( x : [a, b] \to \mathbb{R}^n \) be a \( C^1 \)-path. We say that \( y : [c, d] \to \mathbb{R}^n \) is a \textbf{reparameterisation of} \( x \) if there exists a \textbf{bijective} \( C^1 \)-function \( u : [c, d] \to [a, b] \) so that

\[
y(t) = x(u(t)), \quad t \in [c, d]
\]

Example:

1. Consider the path

\[
x(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad t \in [-\pi/2, \pi/2]
\]

Define

\[
y(t) = \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}, \quad t \in [-\pi/8, \pi/8]
\]

Then \( y \) is a reparameterisation of \( x \): if we define

\( u : [-\pi/8, \pi/8] \to [-\pi/2, \pi/2], \ t \mapsto 4t \)

then \( y(t) = x(u(t)) \). Here \( u^{-1} : [-\pi/2, \pi/2] \to [-\pi/8, \pi/8], u^{-1}(s) = s/4 \) is the inverse function of \( u \).

2. The path

\[
z(t) = \begin{bmatrix} \sqrt{1-t^2} \\ t \end{bmatrix}, \quad t \in [-1, 1]
\]

is a reparameterisation of \( x(t) \): if we define

\( u : [-1, 1] \to [-\pi/2, \pi/2], \ t \mapsto \arcsin(t) \)

then

\[
x(u(t)) = \begin{bmatrix} \cos(\arcsin(t)) \\ \sin(\arcsin(t)) \end{bmatrix} = \begin{bmatrix} \sqrt{1-t^2} \\ t \end{bmatrix} = z(t)
\]

Here we use that if \( s = \arcsin(t) \) then \( \sin(s) = t \) and we have the triangle
3. Let \( x : [a, b] \to \mathbb{R}^n \) be a \( C^1 \)-path. Define the \textbf{opposite path} \( x_{\text{opp}} : [a, b] \to \mathbb{R}^n \) to be

\[
x_{\text{opp}}(t) \overset{\text{def}}{=} x(a + b - t)
\]

\( x_{\text{opp}}(t) \) is a reparameterisation of \( x \); we have

\[
u : [a, b] \to [a, b], \ t \mapsto a + b - t
\]

Observe that \( x_{\text{opp}}(a) = x(b) \) and \( x_{\text{opp}}(b) = x(a) \). The path \( x_{\text{opp}}(t) \) parameterises the same curve as \( x(t) \) but with \textbf{opposite direction}.

\textbf{Important observations:}

- If \( y \) is a reparameterisation of \( x \) then \( x \) is a reparameterisation of \( y \);
- If \( y \) is a reparameterisation of \( x \) then \( y \) and \( x \) are parameterisations of the same curve.

Suppose that \( y : [c, d] \to \mathbb{R}^n \) is a reparameterisation of \( x : [a, b] \to \mathbb{R}^n \), so that

\[
y(t) = x(u(t)).
\]

We say that \( y \) is

- \textbf{orientation-preserving} if \( u(c) = a \) and \( u(d) = b \),
- \textbf{orientation-reversing} if \( u(c) = b \) and \( u(d) = a \).

We now investigate the effects of reparameterisation on vector line integrals.

Let \( F \) be a continuous vector field on \( X \subset \mathbb{R}^n \). Suppose \( y \) is a reparameterisation of \( x \) (with same notation as above), and their (common) image curve is contained in \( X \). Then,

\[
\int_y F \cdot ds = \int_c^d F(y(t)) \cdot y'(t)dt
\]

\[
= \int_c^d F(x(u(t))) \cdot (x'(u(t))u'(t)) dt, \quad \text{because } y = x \circ u
\]

\[
= \begin{cases} 
\int_a^b F(x(u)) \cdot x'(u)du, & \text{if } y \text{ is orientation preserving} \\
\int_b^a F(x(u)) \cdot x'(u)du, & \text{if } y \text{ is orientation reversing}
\end{cases}
\]

\[
= \begin{cases} 
\int \_\_ F \cdot ds, & \text{if } y \text{ is orientation-preserving} \\
- \int \_\_ F \cdot ds, & \text{if } y \text{ is orientation-reversing}
\end{cases}
\]

In words,

- Vector line integrals are independent of orientation-preserving reparameterisations.
- Vector line integrals are independent (up to a sign) of orientation-reversing reparameterisations.
This allows us to define vector line integrals of vector fields $\overrightarrow{F}$ along oriented curves $C$

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{s}$$

rather than along paths.

**Notation:** It is common to denote the vector line integral of $\overrightarrow{F} = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$ along $C$

$$\int_C u(x, y)dx + v(x, y)dy$$

**Example:** Consider the oriented curve $C$ defined as the portion of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

lying in the $y \geq 0$ half-plane oriented clockwise.

We can parameterise the oriented curve $C$ as

$$\overrightarrow{z}(t) = \begin{bmatrix} 3\cos(t) \\ 2\sin(t) \end{bmatrix}, \quad t \in [0, \pi]$$

The importance of what we have shown above is that, if $\overrightarrow{F}$ is a (continuous) vector field on $\mathbb{R}^2$ then

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{s} \overset{\text{def}}{=} \int_x \overrightarrow{F} \cdot d\overrightarrow{s}$$

is independent of how we parameterise the oriented curve $C$: we could have parameterised $C$ by the path

$$\overrightarrow{y}(t) = \begin{bmatrix} t \\ \sqrt{4-4t^2/9} \end{bmatrix}, \quad t \in [-3, 3]$$

to compute the vector line integral of $\overrightarrow{F}$ along $C$.

**Exercise:** Compute

$$\int_x \overrightarrow{F} \cdot d\overrightarrow{s}$$

where $\overrightarrow{F} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$. Now, try to compute the same vector line integral of $\overrightarrow{F}$ along $C$ using the parameterisation $\overrightarrow{y}$:

$$\int_y \overrightarrow{F} \cdot d\overrightarrow{s}$$

*This last line integral might be a bit challenging!*