International Journal of Mathematical Education in Science & Technology
Vol. , No. , , 1–7

Paths and the geometry of l'Hôpital's Rule

Frank J. Swenton, Middlebury College, Middlebury, VT 05753

()

In this short note, we present an easy and instructive proof of l'Hôpital's Rule in both the 0/0 and ∞/∞ cases, obtained by translating the theorem into a simple geometric statement about paths in the plane.

Keywords: l'Hôpital's Rule, path, proof

AMS Subject Classification: 26A06

1 Introduction

During our introduction to differential calculus, we encounter a small set of mathematical ideas that drive our understanding of the subject; from limits, continuity, and the Intermediate Value Theorem to the derivative, Rolle's Theorem, and the Mean Value Theorem, we see basic notions whose logical statements reflect our geometric intuition. Each of these notions can be visualized via a single diagram that drives our understanding and helps us to gain insight into the formal mathematics of the relevant definitions and proofs. This is the ideal toward which we strive when studying any abstract object: a simple working model to mentally grasp as a whole that helps us to understand the object and fill in its logical details.

What goes awry when we step up to face l'Hôpital's Rule? Calculus texts most often prove l'Hôpital's Rule only in the $0/0$ case, often assuming extra hypotheses in order to simplify the argument; the proof of the ∞/∞ is treated in a nonparallel manner or, more often, left to "more advanced texts." The heuristic arguments are incomplete, and the complete arguments are disjointed or overly technical. In each case, the proof of the theorem and its intuitive explanation are disconnected, leaving us wanting for a more unified and complete understanding of l'Hôpital's Rule.

In this short note, we interpret l'Hôpital's Rule as a basic result about paths in the plane. This perspective affords us a proof that treats the $0/0$ and ∞/∞ cases in a parallel fashion and whose entire logic is fully contained in a few simple diagrams. Thus, we obtain a unified argument for both cases that serves simultaneously both to prove and to enlighten.

2 l'Hôpital's Rule

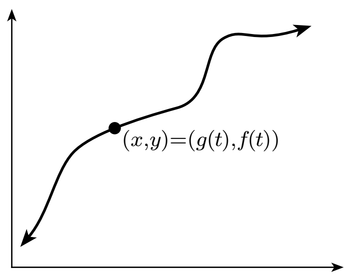
l'Hôpital's Rule. *Let f and g be real-valued functions; let α be any of $a^\pm$, a^+, a^-, $+\infty$, or $-\infty$; and let L be either a real number, $+\infty$, or $-\infty$.*

$$\text{If } \lim_{t\to\alpha} \frac{f'(t)}{g'(t)} = L \text{ and either (i) } \lim_{t\to\alpha} g(t) = \infty$$

$$\text{or (ii) } \lim_{t\to\alpha} f(t) = 0 = \lim_{t\to\alpha} g(t),$$

$$\text{then } \lim_{t\to\alpha} \frac{f(t)}{g(t)} = L \text{ as well.}$$

In order to *see* what this theorem says, we view it as a statement about paths in the plane by setting $(x, y) = (g(t), f(t))$. This allows us to turn the hypotheses and conclusion of l'Hôpital's Rule into statements about easily recognizable mathematical quantities: $f'(t)/g'(t) = \frac{dy}{dt}/\frac{dx}{dt} = \frac{dy}{dx}$ and $f(t)/g(t) = \frac{y}{x}$.



- $\dfrac{f'(t)}{g'(t)} = \dfrac{dy}{dx}$
- $\dfrac{f(t)}{g(t)} = \dfrac{y}{x}$

Taking $t \to \alpha$ as implicit, we can then restate l'Hôpital's Rule as follows:

l'Hôpital's Rule (path version). *Suppose that $(x(t), y(t))$ is a path in the plane. If $\frac{dy}{dx} \to L$ and either $(x, y) \to (0, 0)$ or $x \to \infty$, then $\frac{y}{x} \to L$ as well.*

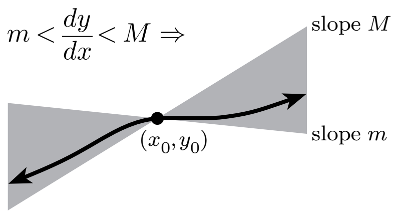
As a first step toward proving this theorem, we examine the quantities $\frac{dy}{dx}$ and $\frac{y}{x}$ as they relate to paths in the plane.

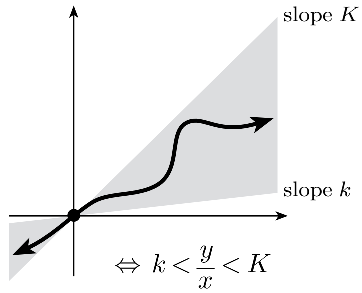
3 The quantities $\frac{dy}{dx}$ and $\frac{y}{x}$

We consider a path $(x(t), y(t))$ to describe a point moving in the plane or the curve traced by this point, as usual. For such a path, $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt}$ tells us the slope at which we're moving along the path.

Now, suppose that we fix some point (x_0, y_0) on our path and know that $\frac{dy}{dx}$ stays between m and M, i.e., that we have some bounds on how steeply the path can climb from (x_0, y_0). After a bit of reflection, we see that it forces all points (x, y) on the path to lie between the lines of slope m and M through our starting point, as in the left-hand figure below.[1]

[1] To prove this, draw a line from the point (x_0, y_0) to any other point (x, y) of the path and apply the Generalized Mean Value Theorem.





The quantity $\frac{y}{x} = \frac{y-0}{x-0}$ is also a slope—namely, the slope from the origin to the point (x, y). Thus, what it means for $\frac{y}{x}$ to stay between k and K is that the each point (x, y) of the path lies between the lines of slope k and K through the origin, as in the right-hand figure above.[2] As the shapes of the two diagrams above suggest, the quantities $\frac{dy}{dx}$ and $\frac{y}{x}$ are related, providing us with the logical link at the core of l'Hôpital's Rule.

4 A geometric proof of l'Hôpital's Rule

We now prove the path version of l'Hôpital's Rule. In light of our above discussion, we treat limits of these quantities geometrically, allowing us to *see* what the hypotheses and conclusion mean. Recall the statement of our theorem: as we follow the path $(x(t), y(t))$, if $\frac{dy}{dx} \to L$ and either $(x, y) \to (0, 0)$ or $x \to \infty$, then $\frac{y}{x} \to L$ as well.

What we need to show is that $\frac{y}{x} \to L$, i.e., that eventually $\frac{y}{x}$ settles as near L as we like. From our earlier discussion of this quantity, $\frac{y}{x}$ being near L means that the point (x, y) lies within a certain cone, as in the left-hand figure below. Thus, we can precisely phrase what we need to show geometrically: given any cone around the line at slope L emanating from the origin, the points (x, y)

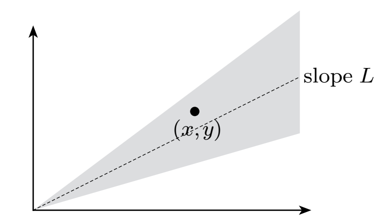
[2]To prove this, just solve the inequality $k < \frac{y}{x} < K$ for y.

eventually settle within that target cone.

Our primary tool for proving this will be the hypothesis that $\frac{dy}{dx} \to L$; this means that as we move along our path, $\frac{dy}{dx}$ eventually settles as near L as we like. As we saw in the previous section, $\frac{dy}{dx}$ being near L allows us to conclude that all subsequent points lie within a certain cone, as in the right-hand figure below. Specifically: for each point (x, y) along our path, all subsequent points will be trapped within a cone of lines around the line at slope L emanating from (x, y). Because $\frac{dy}{dx} \to L$, we are assured that as our point moves along the path, this cone will eventually become as thin as we like.

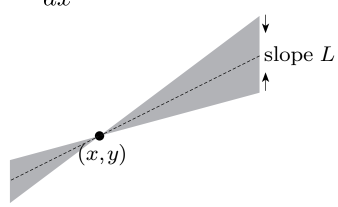
What we need to show:

$\dfrac{y}{x}$ eventually settles close to L.



What we know:

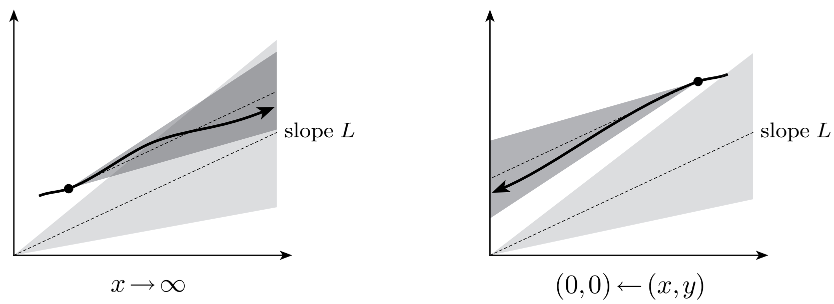
$\dfrac{dy}{dx}$ approaches L.



We can now prove our two cases quite quickly, via the same basic approach in each case; imagine the left- and right-hand figures above superimposed as our point moves along the path. To prove that $\frac{y}{x} \to L$, we are given a [lighter] target cone emanating from the origin, and we need to show that the points of our path eventually settle within it. To do so, we follow our point (x, y) along the path, with our hypothesis that $\frac{dy}{dx} \to L$ telling us that the subsequent points of the path will be trapped within an ever-thinner [darker] cone emanating from its current position. In particular, this cone of known position will eventually become strictly thinner than the the target cone that was given. In each case, we will analyze this configuration and very quickly conclude that the points of the path must eventually stay within the target cone, proving the claim that $\frac{y}{x} \to L$, and thereby establishing the theorem.

Case ∞**.** *If* $x \to \infty$ *and* $\frac{dy}{dx} \to L$*, then* $\frac{y}{x} \to L$ *as well.*

Proof Starting in the configuration discussed above, we see that no matter where our point (x, y) ends up, our [darker] cone of known position will be consumed by our [lighter] target cone, for the darker cone is thinner than the lighter one and has a parallel core. Thus, because $x \to \infty$, the point (x, y) will be forced to settle within the target cone, as sketched at the left-hand figure below. □



Case 0**.** *If* $(x, y) \to (0, 0)$ *and* $\frac{dy}{dx} \to L$*, then* $\frac{y}{x} \to L$ *as well.*

Proof Start again in the configuration discussed above, and suppose (for the sake of contradiction) that the point (x, y) were to subsequently escape our [lighter] target cone, as in the right-hand figure above. In this case, the [darker] cone of known position, being again parallel and thinner, must miss the origin, contradicting our hypothesis that $(x, y) \to (0, 0)$. Thus the points (x, y) of our path must subsequently stay within the target cone. □

5 Remarks

Note that the geometric reasoning above, in stopping just short of $\varepsilon/\delta/M$ details, seamlessly handles the $L = \pm\infty$ cases of the proof: in these cases, the core lines of our cones will merely be vertical, and the geometric argument remains unchanged. Thus, this geometric argument is more robust than would be a fully symbolic treatment, with the full symbolic details of this proof easily recoverable from just a few easily understood figures. We have an argument that is short and simple enough to be easily remembered and understood, achieving not only a proof of l'Hôpital's Rule, but full illumination of its logic, as well.