LIMITS OF ORBIFOLD 0-SPECTRA UNDER COLLAPSING

CARLA FARSI, EMILY PROCTOR, AND CHRISTOPHER SEATON

Abstract. We consider the behavior of the eigenvalue spectrum of the Laplacian of a connected sum of two Riemannian orbifolds as one of the orbifolds in the pair is collapsed to a point. We show that the limit of the eigenvalue spectrum of the connected sum equals the eigenvalue spectrum of the other, non-collapsed, orbifold in the pair. In doing so, we prove the existence of a sequence of singular orbifolds whose eigenvalue spectra come arbitrarily close to the eigenvalue spectrum of a manifold, and a sequence of manifolds whose eigenvalue spectra come arbitrarily close to the eigenvalue spectrum of a singular orbifold.

Contents

1. Introduction

In a series of articles ([3, 5, 4, 28, 6]), Anné, Colbois, and Takahashi, working alone or in pairs, have studied the behavior of eigenvalues of the Laplacian acting on functions and forms on closed orientable manifolds under conditions of collapsing a subset. The authors have considered handled manifolds, manifolds with balls removed, and connected sums of manifolds. They showed that as the handles collapse, the balls fill in, or one part of the connected sum collapses, the limit of the spectrum of the Laplacian acting on functions (resp. p-forms) is equal to the 0-spectrum (resp. p-spectrum) of the Laplace spectrum of the limit space, with careful counting of the zeros of the spectrum.

In particular, in [28], Takahashi proved that for a connected sum $M_1 \sqcup M_2$ of two manifolds, when one collapses M_2 to a point, the 0-spectrum of the connected

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sum converges to that of M_1. The current work generalizes Takahashi's result to the category of orbifolds. Our primary motivation for considering this generalization is related to the well-studied yet open question of whether the spectrum of the Laplacian acting on functions can detect the existence of singularities. In other words, it is unknown whether or not there can be a manifold that is isospectral to an orbifold that has nontrivial singularities. Doyle and Rossetti ([12]) and Rossetti, Schueth, and Weilandt ([24]) have produced examples of isospectral pairs of orbifolds have different maximal isotropy orders. This indicates at least the possibility of an isospectral manifold-orbifold pair. We note that in the same direction, Gordon and Rossetti showed in [17] that it is possible to have a $2p$-dimensional manifold that is isospectral on p-forms to an orbifold with a nontrivial singular set.

In certain contexts, however, it has been shown that a manifold cannot be 0-isospectral to a singular orbifold. In [14], Dryden and Strohmaier showed that isospectral orientable hyperbolic orbisurfaces have the same number of cones points of a particular order. Linowitz and Meyer have also shown that under mild assumptions, there is no such manifold-orbifold pair within the class of lengh-commensurable compact locally symmetric spaces of nonpositive curvature associated to simple Lie groups ([18]). More generally, by the work of Dryden, Gordon, Greenwald and Webb it is impossible for an even- (resp. odd-) dimensional orbifold having an odd- (resp. even-) dimensional singular stratum to be isospectral to a manifold ([13]). In [17], Gordon and Rossetti showed that a manifold and a nonsingular orbifold having a common Riemannian covering cannot be isospectral. Finally, Sutton has shown that it is even that case that if a singular orbifold O and a manifold M admit isospectral Riemannian coverings M_1 and M_2, then O and M cannot be isospectral ([27]).

Here, we show that for the connected sum of two orbifolds O_1 and O_2 as defined in Section 1.1, when O_2 is collapsed to a point, the 0-spectrum of the connected sum converges to the 0-spectrum of O_1. By choosing one of the O_i to be a smooth manifold and approximating the singular metric on the connected sum with a smooth orbifold metric, this generalization of Takahashi's construction yields a sequence of singular orbifolds whose spectra converge to that of a smooth manifold, and similarly a sequence of smooth manifolds whose spectra converge to that of a singular orbifold; see Theorems 1.3 and 1.4. Hence, while the results of this paper do not provide the existence of a manifold that is isospectral to a singular orbifold, we demonstrate that the 0-spectra of a nontrivial orbifold and smooth manifold can in some sense be arbitrarily near one another. In a future paper, the authors will investigate these questions with respect to the behavior of the p-spectrum of the Laplacian.

1.1. **The construction.** In this section, we describe our connected sum construction in order to state our main results. Further details about the Laplace spectrum of the connected sum will be provided in Section 2. We refer the reader to [1] or [20] for more background on Riemannian orbifolds.

Let (O_i, g_i), $i = 1, 2$, be oriented, closed Riemannian orbifolds, both of dimension $n \geq 2$. After possibly scaling g_1 or g_2, we may assume for both (O_1, g_1) and (O_2, g_2) that the injectivity radius is greater than 2. For a point $x_i \in O_i$, denote by $B(x_i, r)$ the ball of radius r about $x_i \in O_i$. We note that here, distance in O_i is measured by the respective metric g_i. Choose a point $p_1 \in O_1$ such that p_1 either is not singular or is an isolated singular point. Choose a point p_2 in O_2 such that the boundary of the $B(p_2, 1)$ contains no singular points. We assume for simplicity that within $B(p_i, 2)$, the metric g_i is Euclidean (see [5, p. 548]). Again, by rescaling, we may further assume that for $i = 1, 2$, the ball $B(p_i, 2)$ is completely contained within a single orbifold chart centered at p_i for $i = 1, 2$.

In what follows, we will consider the connected sum of O_1 and O_2. Our ultimate aim is to shrink O_2, so we use the following construction to make this simple. Let $O_i(r) = O_i \backslash B(p_i, r)$. Now, suppose that $\varepsilon < 1$. If $S^{n-1}(\varepsilon) \subset \mathbb{R}^n$ denotes the sphere of radius ε with the standard metric h_ε inherited from the Euclidean metric on $\mathbb{R}^n$, then, letting $\partial O_i(\varepsilon)$ denote the boundary of $O_i(\varepsilon)$ with inherited boundary metric ∂g_i, by our hypothesis that g_i is Euclidean on $B(p_i, 2)$, we see that $(\partial O_1(\varepsilon), \partial g_1)$ is isometric to $(S^{n-1}(\varepsilon), h_\varepsilon)$ and $(\partial O_2(1), \varepsilon^2 \partial g_2)$ is isometric to $(S^{n-1}(1), \varepsilon^2 h_1)$. Furthermore, $(S^{n-1}(\varepsilon), h_\varepsilon)$ can be mapped isometrically to $(S^{n-1}(1), \varepsilon^2 h_1)$ via the restriction of the map $\mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto \varepsilon^{-1} x$. Taking the composition of these maps, $\varphi_\varepsilon : (\partial O_1(\varepsilon), \partial g_1) \to (\partial O_2(1), \varepsilon^2 \partial g_2)$, allows us to create the connected sum of (O_1, g_1) and $(O_2, \varepsilon^2 g_2)$ by excising balls about p_1 and p_2 and identifying the boundaries of $(O_1(\varepsilon), g_1)$ and $(O_2(1), \varepsilon^2 g_2)$ via φ_ε so that the resulting sum is oriented. Call this connected sum (O, g_ε), where

$$g_\varepsilon = \begin{cases} g_1 & \text{on } O_1(\varepsilon) \\ \varepsilon^2 g_2 & \text{on } O_2(1). \end{cases}$$

Under this construction, (O, g_ε) is a closed, smooth orbifold, but the metric is not smooth along the glued boundary. Nevertheless, letting $\mathcal{C}_b^\infty(O, g_\varepsilon)$ denote the algebra of pairs of smooth functions on $O_1(\varepsilon)$ and $O_2(1)$ with appropriate boundary conditions, see Equation (2.1), we can for each $0 < \varepsilon \ll 1$ define a Laplacian Δ_ε acting on $\mathcal{C}_b^\infty(O, g_\varepsilon)$; see Section 2. The spectrum of Δ_ε is a discrete sequence

$$0 = \lambda_0(O, g_\varepsilon) < \lambda_1(O, g_\varepsilon) \leq \lambda_2(O, g_\varepsilon) \leq \cdots \to \infty.$$

1.2. **The results.** In what follows, we study the behavior of the spectrum of (O, g_ε) as $\varepsilon \to 0$. In particular, we will show the following.

Theorem 1.1. *For $0 < \varepsilon \ll 1$, let (O, g_ε) and its Laplace spectrum be constructed as above. Suppose that (O_1, g_1) has Laplace spectrum $0 = \lambda_0(O_1, g_1) < \lambda_1(O_1, g_1) \leq \lambda_2(O_1, g_1) \leq \cdots$. Then for all $j = 0, 1, \ldots$,*

$$\lim_{\varepsilon \to 0} \lambda_j(O, g_\varepsilon) = \lambda_j(O_1, g_1).$$

For each integer $k > 0$, the convergence is uniform for $j = 0, 1, \ldots, k$.

In Section 5, we will confirm that for the $\mathcal{C}^0$-topology of metrics on an orbifold, eigenvalues vary continuously with the metric. This will allow us to prove the following theorem.

Theorem 1.2. *Suppose that $\eta > 0$ and let $k > 0$ be any integer. Let O be the connected sum described above. There is a smooth metric $g_{\eta,k}$ on O depending only on η and k such that for all $j = 0, 1, \ldots, k$,*

$$|\lambda_j(O, g_{n,k}) - \lambda_j(O_1, g_1)| < \eta.$$

By taking (O_1, g_1) to be a Riemannian manifold and (O_2, g_2) to be a singular orbifold (and recalling that the singular set of O_2 lies outside $B(p_2, 1)$), the following theorem follows from Theorem 1.2.

Theorem 1.3. *There is a sequence* $\{(O_i, g_i)\}_{i\in\mathbb{N}}$ *of smooth singular orbifolds such that as* $i \to \infty$*, the spectra of* (O_i, g_i) *tend to the spectrum of a manifold* (M, g) *in the sense that for any* $\eta > 0$ *and any integer* $k > 0$*,*

$$|\lambda_j(O_i, g_i) - \lambda_j(M, g)| < \eta$$

for all $j = 0, 1, \ldots, k$*.*

Finally, let us note that for every $n \geq 1$, there is a compact $2n$-dimensional orbifold with a single singular point. When $n = 1$, the well-known teardrop is such an example. To see this for $n \geq 2$, let $s, a_1, \ldots, a_t$ be positive integers such that $s \geq 2$ and $\gcd(s, a_i) = 1$ for each i. Recall that the *lens space* $L(t, a_1 \ldots, a_t)$ is the quotient of $S^{2t+1} \subset \mathbb{C}^{t+1}$ by the action of $\mathbb{Z}_s$ generated by $(z_1, \ldots, z_{t+1}) \mapsto (\zeta^{a_1} z_1, \ldots, \zeta^{a_t} z_t, \zeta z_{t+1})$ where ζ is a primitive sth root of unity. By [25, Theorems 4.4, 4.10, and 4.16], there exists for each $t \geq 1$ a lens space $L(t, a_1 \ldots, a_t)$ that is the boundary of an oriented manifold M (that is compact by construction). Letting $B(t, a_1 \ldots, a_t)$ denote the orbifold given by the quotient of the closed unit ball $\mathbb{B}^{2t+2} \subset \mathbb{C}^{t+1}$ by the same action and noting that $B(t, a_1 \ldots, a_t)$ has a single singular point with isotropy $\mathbb{Z}_s$, we have $\partial B(t, a_1 \ldots, a_t) = L(t, a_1 \ldots, a_t)$. Hence, identifying ∂M with $\partial B(t, a_1 \ldots, a_t)$ yields a closed, oriented orbifold with a single singular point as claimed.

With this, suppose that (O_1, g_1) is an oriented orbifold with a single singular point p_1. If we excise a neighborhood about p_1 and take the connected sum with a manifold (M_2, g_2), we obtain the following.

Theorem 1.4. *For each* $n \geq 2$*, there is a sequence* $\{(M_i, g_i)\}_{i\in\mathbb{N}}$ *of smooth manifolds of dimension* $2n$ *such that as* $i \to \infty$*, the spectra of* (M_i, g_i) *tend to the spectrum of an orbifold* (O, g) *in the sense that for any* $\eta > 0$ *and for any integer* $k > 0$*,*

$$|\lambda_j(M_i, g_i) - \lambda_j(O, g)| < \eta$$

for all $j = 0, 1, \ldots, k$*.*

1.3. **Summary of the paper.** In Section 2, we define the Laplacian acting on functions on the connected sum (O, g_ε) constructed above and confirm that it has a discrete eigenvalue spectrum that tends to infinity. Sections 3 and 4 together constitute a proof of Theorem 1.1. In Section 3, we show that for each k, $\limsup_{\varepsilon\to 0} \lambda_k(O, g_\varepsilon) \leq \lambda_k(O_1, g_1)$ and in Section 4, we show that for each k, $\liminf_{\varepsilon\to 0} \lambda_k(O, g_\varepsilon) \geq \lambda_k(O_1, g_1)$. Finally, in Section 5 we prove that in our context, the eigenvalues $\lambda_k(O, g_\varepsilon)$ vary continuously with respect to the metric, allowing us to obtain Theorem 1.2.

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2. The Laplacian on the connected sum

In this section, we define the Laplacian Δ_ε acting on functions on the connected sum (O, g_ε) defined in Section 1.1. We confirm that Δ_ε has a discrete spectrum of eigenvalues, each having only finite multiplicity, tending to infinity.

The orbifold that we constructed in Section 1 was the connected sum of two orbifolds with boundary. In order to carefully define an orbifold with boundary, we begin by defining a real half space. Our treatment here follows that of [1], [10], and [26].

Given a nonzero vector $u \in \mathbb{R}^n$, the corresponding half space $\mathbb{R}^n_u$ is given by

$$\mathbb{R}^n_u := \{x \in \mathbb{R}^n \,|, \langle x, u\rangle \geq 0\}.$$

We say that a map $h : \mathbb{R}^n_u \to \mathbb{R}^n$ is differentiable at a boundary point of $\mathbb{R}^n_u$ if it has a differentiable extension $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$. The derivative Dh of h is the restriction of $D\tilde{h}$ to $\mathbb{R}^n_u$.

Definition 2.1. Let X_K be a second countable Hausdorff space and let U be an open set in X_K. An *orbifold chart* over U is a triple $(\tilde{U}, \Gamma_U, \pi_U)$ satisfying

(1) $\tilde{U}$ is a connected open set of $\mathbb{R}^n_u$ for some nonzero $u \in \mathbb{R}^n$,
(2) Γ_U is a finite group acting by diffeomorphisms on $\tilde{U}$,
(3) $\pi_U : \tilde{U} \to U$ is a continuous map such that $\pi_U\gamma = \pi_U$ for all $\gamma \in \Gamma_U$ and which induces a homeomorphism from $\tilde{U}/\Gamma_U$ onto U.

Given two orbifold charts $(\tilde{U}, \Gamma_U, \pi_U)$ and $(\tilde{V}, \Gamma_V, \pi_V)$ with $U \subseteq V \subseteq X_K$, an *embedding* $\lambda : (\tilde{U}, \Gamma_U, \pi_U) \hookrightarrow (\tilde{V}, \Gamma_V, \pi_V)$ is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ with $\pi_V\lambda = \pi_U$.

Definition 2.2. An n-dimensional *orbifold with boundary* K is a second countable Hausdorff space X_K, called the underlying space of K, covered by a maximal atlas of orbifold charts $\{(U_a, \Gamma_{U_a}, \pi_{U_a})\}_{a\in A}$ satisfying the following compatibility condition: for any two charts $(\tilde{U}_a, \Gamma_{U_a}, \pi_{U_a})$ and $(\tilde{U}_b, \Gamma_{U_b}, \pi_{U_b})$ and point $x \in U_a \cap U_b$ there is an open neighborhood $U_c \subset U_a \cap U_b$ about x and a chart $(\tilde{U}_c, \Gamma_{U_c}, \pi_{U_c})$ over U_c such that there are smooth embeddings $\lambda_a : (\tilde{U}_c, \Gamma_{U_c}, \pi_{U_c}) \hookrightarrow (\tilde{U}_a, \Gamma_{U_a}, \pi_{U_a})$ and $\lambda_b : (\tilde{U}_c, \Gamma_{U_c}, \pi_{U_c}) \hookrightarrow (\tilde{U}_b, \Gamma_{U_b}, \pi_{U_b})$. The *boundary* ∂K of K is the set of all points $p \in K$ for which there is a chart $(\tilde{U}_a, \Gamma_{U_a}, \pi_{U_a})$ such that $\langle \tilde{p}, u_a\rangle = 0$ for all points $\tilde{p} \in \pi^{-1}_{U_a}(p)$.

The compatibility condition for orbifold charts allows us to consider a form of transition function between overlapping charts $(\tilde{U}_a, \Gamma_{U_a}, \pi_{U_a})$ and $(\tilde{U}_b, \Gamma_{U_b}, \pi_{U_b})$, namely $\lambda_b\lambda_a^{-1}$ (see [1, p.10]). With transition functions in hand, we can define orientability.

Definition 2.3. We say that an orbifold K is *oriented* if it is covered by an atlas of orbifold charts such that for all pairs $(\tilde{U}_a, \Gamma_{U_a}, \pi_{U_a})$ and $(\tilde{U}_b, \Gamma_{U_b}, \pi_{U_b})$, $\det D(\lambda_b\lambda_a^{-1}) > 0$.

Given an orbifold K, with or without boundary, we can construct a Riemannian structure on K by placing a Γ_U-invariant Riemannian metric on the local cover $\tilde{U}$ of each orbifold chart $(\tilde{U}, \Gamma_U, \pi_U)$ and patching these local metrics together with a partition of unity. A smooth orbifold with a Riemannian structure is called a *Riemannian orbifold* (see [10, 16]).

Definition 2.4. Let (K, g) be an oriented Riemannian orbifold with boundary. Suppose that there are no singular points on the boundary ∂K of K. Let N be the inward pointing unit normal vector field along ∂K. We say that K is *collared* if there exists a smooth diffeomorphism $\Sigma : \partial K \times [0, 1) \to K$ onto an open neighborhood of ∂K in K such that $\Sigma(p, 0) = p$ and $D\Sigma_{(p,0)}(0, 1) = N_p$ for all $p \in \partial K$.

For the remainder of the paper, we assume that all Riemannian orbifolds are compact and oriented. If the orbifold has boundary, we assume that there are no singular points on the boundary and that the orbifold is collared. When we glue two orbifolds with boundary together, as in Section 1.1, we choose orientations on the two orbifolds in such a way that the connected sum has a natural orientation.

Definition 2.5. Let (K, g) be a compact Riemannian orbifold, with or without boundary. Suppose that $\{(\tilde{U}_i, \Gamma_{U_i}, \pi_{U_i})\}_{i=1}^m$ is a finite covering of (K, g) by orbifold charts. Let $\{\rho_i\}_{i=1}^m$ be a partition of unity subordinate to $\{U_i\}_{i=1}^m$. For a function $f \in \mathcal{C}^\infty(K)$ define the integral of f over K by

$$\int_K f\, dv_g := \sum_{i=1}^m \frac{1}{|\Gamma_{U_i}|} \int_{\tilde{U}_i} \tilde{\rho}_i(\tilde{x}) \tilde{f}(\tilde{x})\, d\tilde{v}_{\tilde{g}},$$

where $\sim$ denotes the lift in all cases and dv_g denotes the volume form with respect to g. We note that although the definition makes use of a particular covering of K, it can be shown to be independent of the choice of covering (see [10], [16] and [1, p.34]).

As pointed out by Chiang [10, p.320], given a smooth function f on (K, g), the lift of f to any orbifold chart $\tilde{U}$ is Γ_U-invariant, and thus the gradient ∇f is Γ_U-invariant on $\tilde{U}$ as well. Therefore, ∇f is well-defined on (K, g).

Sobolev spaces for closed orbifolds were introduced in [10] and [16]. Although both authors gave the definition for orbifolds without boundary, the definition generalizes readily to compact orbifolds with boundary. (See [26] for a careful discussion of Sobolev spaces for manifolds with boundary.)

Definition 2.6. Let (K, g) be a compact Riemannian orbifold with boundary. Suppose that s is a nonnegative integer. The Sobolev space $H^s(K, g)$ is the completion of $\mathcal{C}^\infty(K)$ with respect to the norm

$$\|f\|_{H^s(K,g)} = \left(\sum_{l=0}^s \int_K |\nabla^l f|^2\, dv_g \right)^{\frac{1}{2}}.$$

We remark that $H^0(K, g) = L^2(K, g)$ and that the H^1-norm is given by

$$\|f\|_{H^1(K,g)} = \left(\int_K f^2\, dv_g + \int_K (\nabla f)^2\, dv_g \right)^{\frac{1}{2}} = \left(\langle f, f\rangle_{L^2(K,g)} + \langle \nabla f, \nabla f\rangle_{L^2(K,g)} \right)^{\frac{1}{2}}.$$

Also, as usual, $H^s(K, g) \subset H^t(K, g)$ when $s > t$ and the following orbifold version of Rellich's theorem holds (see [10, 16]).

Theorem 2.7 (Rellich's Theorem)**.** *For (K, g) a compact orbifold without boundary and $s > t$, the inclusion $H^s(K, g)$ into $H^t(K, g)$ is compact.*

We now define Sobolev spaces for the connected sum (O, g_ε) constructed in Section 1.1. These Sobolev spaces will ultimately allow us to define the Laplacian Δ_ε on (O, g_ε) in such a way that Δ_ε is a non-negative self-adjoint operator whose spectrum is discrete and tending to infinity.

Definition 2.8. For the connected sum (O, g_ε), define $L^2(O, g_\varepsilon)$ by

$$L^2(O, g_\varepsilon) := L^2(O_1(\varepsilon), g_1) \oplus L^2(O_2(1), \varepsilon^2 g_2).$$

with the componentwise inner product. In this way, $L^2(O, g_\varepsilon)$ is a Hilbert space.

Remark 2.9. Recall that by the trace theorem for manifolds with boundary, there are trace operators from $H^1(M,g)$ to $L^2(\partial M, \partial g)$ and from $H^2(M,g)$ to $H^1(\partial M, \partial g)$ satisfying $\|u_{|\partial M}\|_{L^2(\partial M,\partial g)} \leq C\|u\|_{H^1(M,g)}$ for some choice of C independent of u (see [15, p.258] and [29, Section 4.4]). The trace theorem for manifolds with boundary is proven by patching together local trace operators between Sobolev spaces on regions in $\mathbb{R}^n$ and their boundaries. Since we consider collared orbifolds with no singularities along the boundary, the proof of the trace thereom carries directly to our setting. Thus, the following definitions make sense.

Definition 2.10. The first and second Sobolev spaces on (O, g_ε) are defined as follows.

$$H^1(O, g_\varepsilon) := \{f = (f_1, f_2) \in H^1(O_1(\varepsilon), g_1) \oplus H^1(O_2(1), \varepsilon^2 g_2) \mid \\ f_{1_{|\partial O_1(\varepsilon)}} = f_{2_{|\partial O_2(1)}} \circ \varphi_\varepsilon \text{ in } L^2(\partial O_1(\varepsilon), \partial g_1)\}$$

$$H^2(O, g_\varepsilon) := \{f = (f_1, f_2) \in H^2(O_1(\varepsilon), g_1) \oplus H^2(O_2(1), \varepsilon^2 g_2) \mid \\ f_{1_{|\partial O_1(\varepsilon)}} = f_{2_{|\partial O_2(1)}} \circ \varphi_\varepsilon \text{ in } H^1(\partial O_1(\varepsilon), \partial g_1), \\ (\nu_1 \cdot f_1)_{|\partial O_1(\varepsilon)} = -(\varepsilon^{-1}\nu_2 \cdot f_2)_{|\partial O_2(1)} \circ \varphi_\varepsilon \text{ in } L^2(\partial O_1(\varepsilon), \partial g_1)\}$$

where ν_1 and ν_2 denote the outward unit normal vector fields along $(\partial O_1(\varepsilon), \partial g_1)$ and $(\partial O_2(1), \partial g_2)$, respectively, and thus $\varepsilon^{-1}\nu_2$ is the unit outward normal on $(\partial O_2(1), \varepsilon^2 g_2)$. The inner product on each space is given by the direct sum of the inner products on the two component pieces.

We have the following version of Rellich's theorem for the connected sum (O, g_ε).

Theorem 2.11. *For $L^2(O, g_\varepsilon)$ and $H^1(O, g_\varepsilon)$ defined as above, the inclusion of $H^1(O, g_\varepsilon)$ into $L^2(O, g_\varepsilon)$ is compact.*

Proof. By assumption, the orbifolds $(O_1(\varepsilon), g_1)$ and $(O_2(1), \varepsilon^2 g_2)$ that we use to construct (O, g_ε) are collared orbifolds with no singularities along the boundary. Thus, using an argument similar to [29, p.333], we may double along the boundaries of $O_1(\varepsilon)$ and $O_2(1)$ and use Theorem 2.7 to conclude that $H^1(O_1(\varepsilon), g_1)$ is compactly embedded in $L^2(O_1(\varepsilon), g_1)$ and $H^1(O_2(1), \varepsilon^2 g_2)$ is compactly embedded in $L^2(O_2(1), \varepsilon^2 g_2)$. But the norms on these respective spaces are all computed via a direct sum, and since $H^1(O, g_\varepsilon)$ is a closed subspace of $H^1(O_1(\varepsilon), g_1) \oplus H^1(O_2(1), \varepsilon^2 g_2)$, it follows directly that the inclusion of $H^1(O, g_\varepsilon)$ into $L^2(O, g_\varepsilon)$ is compact. $\square$

We will now define the Laplacian Δ_ε for (O, g_ε). In order to ensure that Δ_ε is a non-negative, self-adjoint operator, we begin by considering

$$\mathcal{C}_b^\infty(O, g_\varepsilon) = \{f = (f_1, f_2) \in \mathcal{C}^\infty(O_1(\varepsilon)) \oplus \mathcal{C}^\infty(O_2(1)) \mid \\ f_{1_{|\partial O_1(\varepsilon)}} = f_{2_{|\partial O_2(1)}} \circ \varphi_\varepsilon, \\ (\nu_1 \cdot f_1)_{|\partial O_1(\varepsilon)} = -(\varepsilon^{-1}\nu_2 \cdot f_2)_{|\partial O_2(1)} \circ \varphi_\varepsilon\}. \tag{2.1}$$

Note that $\mathcal{C}_b^\infty(O, g_\varepsilon)$ is a dense subspace of $L^2(O, g_\varepsilon)$.

Definition 2.12. For $f = (f_1, f_2) \in \mathcal{C}_b^\infty(O, g_\varepsilon)$, define Δ_ε by

$$\Delta_\varepsilon f := (\Delta_{g_1} f_1, \Delta_{\varepsilon^2 g_2} f_2).$$

We use Green's theorem to show that Δ_ε is symmetric and non-negative on $\mathcal{C}_b^\infty(O, g_\varepsilon)$. Recall the following two versions of Green's theorem.

Theorem 2.13 (Green's Theorem)**.** *Let f, h be $\mathcal{C}^\infty$ functions on an orbifold K with boundary ∂K. Then*

(1) $\langle \Delta f, h\rangle_{L^2(K,g)} = \langle \nabla f, \nabla h\rangle_{L^2(K,g)} - \int_{\partial K} h(\nu \cdot f)\, dv_{\partial g}$.

(2) $\langle \Delta f, h\rangle_{L^2(K,g)} - \langle f, \Delta h\rangle_{L^2(K,g)} = \int_{\partial K} (f(\nu\cdot h) - h(\nu \cdot f))\, dv_{\partial g}$,

where ν denotes the unit outward pointing normal on ∂K.

Proof. The manifold version Green's theorem is derived from Stokes' theorem by arguments that are all local in nature, and thus generalize to orbifolds (see, for example, [29, p.154]). Therefore, if Stokes' theorem holds for orbifolds, so does Green's theorem. But the proof of Stoke's theorem is also given locally and patched together with a partition of unity (see [29, p.80]). Therefore it holds for orbifolds and we are done. □

Letting $K = O_1(\varepsilon)$ and $O_2(1)$ respectively in Part (1) of Green's theorem and applying the theorem to a function $f = (f_1, f_2) \in \mathcal{C}_b^\infty(O, g_\varepsilon)$, we can use Green's theorem to confirm that Δ_ε is non-negative on $\mathcal{C}_b^\infty(O, g_\varepsilon)$ as follows:

$$\begin{aligned}
\langle \Delta_\varepsilon f, f\rangle_{L^2(O,g_\varepsilon)} &= \langle \Delta_{g_1} f_1, f_1\rangle_{L^2(O_1(\varepsilon),g_1)} + \langle \Delta_{\varepsilon^2 g_2} f_2, f_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} \\
&= \langle \nabla f_1, \nabla f_1\rangle_{L^2(O_1(\varepsilon),g_1)} - \int_{\partial O_1(\varepsilon)} f_1(\nu_1\cdot f_1)\, dv_{\partial g_1} \\
&\quad + \langle \nabla f_2, \nabla f_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} - \int_{\partial O_2(1)} f_2(\varepsilon^{-1}\nu_2 \cdot f_2)\, dv_{\partial \varepsilon^2 g_2} \\
&= \langle \nabla f_1, \nabla f_1\rangle_{L^2(O_1(\varepsilon),g_1)} + \langle \nabla f_2, \nabla f_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} \\
&\geq 0.
\end{aligned}$$

The last line above follows from the facts that $f \in \mathcal{C}_b^\infty(O, g_\varepsilon)$ and that φ_ε is an isometry so $\varphi_\varepsilon^*(dv_{\varepsilon^2 \partial g_2}) = dv_{\partial g_1}$.

We can also use Green's theorem to confirm that Δ_ε is symmetric on $\mathcal{C}_b^\infty(O, g_\varepsilon)$. Indeed, for functions $f = (f_1, f_2), h = (h_1, h_2) \in \mathcal{C}_b^\infty(O, g_\varepsilon)$,

$$\begin{aligned}
&\langle \Delta_\varepsilon f, h\rangle_{L^2(O,g_\varepsilon)} - \langle f, \Delta_\varepsilon h\rangle_{L^2(O,g_\varepsilon)} \\
&\qquad = \langle \Delta_{g_1} f_1, h_1\rangle_{L^2(O_1(\varepsilon),g_1)} + \langle \Delta_{\varepsilon^2 g_2} f_2, h_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} \\
&\qquad\quad - \langle f_1, \Delta_{g_1} h_1\rangle_{L^2(O_1(\varepsilon),g_1)} - \langle f_2, \Delta_{\varepsilon^2 g_2} h_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} \\
&\qquad = \langle \Delta_{g_1} f_1, h_1\rangle_{L^2(O_1(\varepsilon),g_1)} - \langle f_1, \Delta_{g_1} h_1\rangle_{L^2(O_1(\varepsilon),g_1)} \\
&\qquad\quad + \langle \Delta_{\varepsilon^2 g_2} f_2, h_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} - \langle f_2, \Delta_{\varepsilon^2 g_2} h_2\rangle_{L^2(O_2(1),\varepsilon^2 g_2)} \\
&\qquad = \int_{\partial O_1(\varepsilon)} (f_1(\nu_1 \cdot h_1) - h_1(\nu_1\cdot f_1))\, dv_{\partial g_1} \\
&\qquad\quad + \int_{\partial O_2(1)} (f_2(\varepsilon^{-1}\nu_2 \cdot h_2) - h_2(\varepsilon^{-1}\nu_2\cdot f_2))\, dv_{\partial\varepsilon^2 g_2} \\
&\qquad = 0,
\end{aligned}$$

where we again use the facts that $f, h \in \mathcal{C}_b^\infty(O, g_\varepsilon)$ and $\varphi_\varepsilon^*(dv_{\varepsilon^2 \partial g_2}) = dv_{\partial g_1}$.

Since Δ_ε is non-negative and symmetric on $\mathcal{C}_b^\infty(O, g_\varepsilon)$, if we define a bilinear form q_ε on $\mathcal{C}_b^\infty(O, g_\varepsilon) \times \mathcal{C}_b^\infty(O, g_\varepsilon)$ by

$$q_\varepsilon(f, h) := \langle \Delta_\varepsilon f, h\rangle_{L^2(O,g_\varepsilon)},$$

then the associated quadratic form q'_ε on $\mathcal{C}_b^\infty(O, g_\varepsilon)$ given by $q'_\varepsilon(f) = q_\varepsilon(f, f)$ is closable and its closure is associated with a self-adjoint extension of Δ_ε (see [11, Theorem 4.4.5] and [23, Theorem X.23]). We will denote the extension by Δ_ε as well. We remark that this self-adjoint extension of Δ_ε is the Friedrich's extension and that the domain of the extension is $H^2(O, g_\varepsilon)$. The domain of the corresponding (closed) quadratic form q'_ε is $H^1(O, g_\varepsilon)$.

The following proposition gives an explicit formulation for q_ε.

Proposition 2.14. *For $f = (f_1, f_2) \in H^2(O, g_\varepsilon)$ and $h = (h_1, h_2) \in H^1(O, g_\varepsilon)$ the bilinear form q_ε given by*

$$q_\varepsilon(f, h) = \int_{O_1(\varepsilon)} \langle \nabla f_1, \nabla h_1 \rangle \, dv_{g_1} + \int_{O_2(1)} \langle \nabla f_2, \nabla h_2 \rangle \, dv_{\varepsilon^2 g_2}$$

is induced from the Laplacian as above, i.e.

$$q_\varepsilon(f, h) = \langle \Delta_\varepsilon f, h \rangle_{L^2(O, g_\varepsilon)}.$$

Proof. By using approximation, we can extend the statement of Green's theorem to say that

$$\langle \Delta f, h \rangle_{L^2(K,g)} = \langle \nabla f, \nabla h \rangle_{L^2(K,g)} - \int_{\partial K} h(\nu \cdot f) \, dv_{\partial g}.$$

for $f \in H^2(K, g)$, $h \in H^1(K, g)$ (see [29, Exercise 4.4.2.]). The rest of the proof of the proposition is almost identical to the proof given above that under the appropriate gluing conditions Δ_ε is a non-negative operator. $\square$

Finally, we confirm that the spectrum of the Laplacian Δ_ε has the following familiar properties.

Theorem 2.15. *The spectrum of eigenvalues of the Laplacian Δ_ε is a discrete set $\{\lambda_i\}$ with $\lambda_i \geq 0$ for all i. Each eigenvalue appears in the spectrum only finitely many times, and the sequence $\{\lambda_i\}$ tends to ∞ as $i \to \infty$. There is also a complete orthonormal basis of $L^2(O, g_\varepsilon)$ consisting of eigenfunctions $\{f_i\}_{i=1}^\infty$ of Δ_ε.*

Proof. Using Corollary 4.2.3. in [11], if we can show that the resolvent operator $(\Delta_\varepsilon + 1)^{-1}$ is compact, we will be done. But by Exercise 4.2 also in [11], $(\Delta_\varepsilon + 1)^{-1}$ is compact if and only if $H^1(O, g_\varepsilon)$, with norm

$$\|f\|_{q_\varepsilon} := (q_\varepsilon(f, f) + \|f\|^2_{L^2(O, g_\varepsilon)})^{\frac{1}{2}},$$

is compactly embedded in $L^2(O, g_\varepsilon)$.

By Proposition 2.14, the q_ε-norm is the same as the standard H^1-norm on $H^1(O, g_\varepsilon)$. By Rellich's theorem (Theorem 2.11), $H^1(O, g_\varepsilon)$, with the H^1-norm, is compactly embedded in $L^2(O, g_\varepsilon)$. Thus, the spectrum of Δ_ε has all of the desired properties. $\square$

3. An upper bound for $\lambda_k(O, g_\varepsilon)$ as $\varepsilon \to 0$

In this section, we prove the following proposition, using the min-max principle. For simplicity of notation, throughout this section we denote the L^2-norm and the L^2-inner product on (O_1, g_1) by $\| \cdot \|_0$ and $\langle \cdot, \cdot \rangle_0$ respectively.

Proposition 3.1. *Suppose that (O_1, g_1) and (O, g_ε) are defined as in Section 1.1. Then*

$$\limsup_{\varepsilon \to 0} \lambda_k(O, g_\varepsilon) \leq \lambda_k(O_1, g_1).$$

Remark 3.2. The proof of Proposition 3.1 is the first half of the proof of Theorem 1.1.

We begin by considering the cutoff function $\chi_\varepsilon : [0,\infty) \to [0,1]$ given by

$$\chi_\varepsilon(r) = \begin{cases} 0 & 0 \le r \le \varepsilon \\ -\frac{2}{\ln \varepsilon} \ln\left(\frac{r}{\varepsilon}\right) & \varepsilon \le r \le \sqrt{\varepsilon} \\ 1 & \sqrt{\varepsilon} \le r \end{cases}$$

(see [28, Section 3] and [4, Section 6]). Now, define $\chi_\varepsilon : O_1 \to [0,1]$ by $\chi_\varepsilon(x) = \chi_\varepsilon(d_{g_1}(p_1,x))$.

Remark 3.3. In the case we are considering here, since we are working with a radial distance based at p_1 and orthogonal charts centered at p_1, the distance in O_1 between the center point p_1 and a point x contained in an orbifold chart about p_1 is exactly equal to the distance in the associated manifold chart above p_1 between the center point 0 and any point in the preimage of x.

We now prove two lemmas about χ_ε that will aid us in our proof of Proposition 3.1.

Lemma 3.4. *Let $\chi_\varepsilon : O_1 \to [0,1]$ be defined as above. Suppose that $\{f_0, f_1, \dots, f_k\}$ is a linearly independent collection of smooth functions on (O_1, g_1). There is a value $\tilde{\varepsilon} > 0$ such that for all $\varepsilon < \tilde{\varepsilon}$, the collection $\{\chi_\varepsilon f_0, \chi_\varepsilon f_1, \dots, \chi_\varepsilon f_k\}$ is a linearly independent set.*

Proof. Recall that $O_1(\sqrt{\varepsilon}) = O_1 \backslash B(p_1, \sqrt{\varepsilon})$. For each j and $\varepsilon > 0$, it is clear from the definition of χ_ε that $f_j|_{O_1(\sqrt{\varepsilon})} = \chi_\varepsilon f_j|_{O_1(\sqrt{\varepsilon})}$. As extending each function in linearly independent set of functions preserves linear independence, we will show that $\{\chi_\varepsilon f_0, \chi_\varepsilon f_1, \dots, \chi_\varepsilon f_k\}$ is linearly independent for some $\varepsilon > 0$ by demonstrating that $\{f_0|_{O_1(\sqrt{\varepsilon})}, f_1|_{O_1(\sqrt{\varepsilon})}, \dots, f_k|_{O_1(\sqrt{\varepsilon})}\}$ is linearly independent.

For $0 < \varepsilon < 1$, define the set

$$A_\varepsilon := \left\{ (a_0, a_1, \dots, a_k) \in \mathbb{R}^{k+1} \Big| \sum_{i=0}^{k} a_j f_j|_{O_1(\sqrt{\varepsilon})} = 0 \right\}.$$

Note that if $\varepsilon < \varepsilon'$, then $A_\varepsilon \subset A_{\varepsilon'}$. Assume by way of contradiction that there is a nonzero element of $\bigcap_{\varepsilon\in(0,1]} A_\varepsilon$, say $0 \ne (c_0, \dots, c_k) \in \bigcap_{\varepsilon\in(0,1]} A_\varepsilon$. Then as every $x \in O_1\backslash\{p_1\}$ is an element of $O_1(\sqrt{\varepsilon})$ for some $\varepsilon > 0$, we have that $\sum_{i=0}^k c_j f_j(x) = 0$ for every $x \ne p_1$. As $\sum_{i=0}^k c_j f_j$ is smooth and hence continuous, $\sum_{i=0}^k c_j f_j(p_1) = 0$, and $\sum_{i=0}^k c_j f_j$ is the zero function, contradicting the linear independence of $\{f_0, f_1, \dots, f_k\}$. Therefore, $\bigcap_{\varepsilon\in(0,1]} A_\varepsilon$ contains only the zero tuple. Recalling that $A_\varepsilon \subset A_{\varepsilon'}$ for $\varepsilon < \varepsilon'$, it follows that there is an $\tilde{\varepsilon} \in (0,1]$ such that $A_{\tilde{\varepsilon}}$ contains only the zero tuple, and hence that $\{f_0|_{O_1(\sqrt{\varepsilon})}, f_1|_{O_1(\sqrt{\varepsilon})}, \dots, f_k|_{O_1(\sqrt{\varepsilon})}\}$ is linearly independent for each $\varepsilon < \tilde{\varepsilon}$. □

Recall the q_ε-norm on $H^1(O, g_\varepsilon)$, defined in the proof of Theorem 2.15:

$$\|f\|_{q_\varepsilon} := (q_\varepsilon(f,f) + \|f\|^2_{L^2(O,g_\varepsilon)})^{\frac{1}{2}}.$$

Lemma 3.5. *With $\chi_\varepsilon : O_1 \to [0,1]$ defined as above,*

$$\lim_{\varepsilon\to 0} \|\chi_\varepsilon - 1\|_{q_\varepsilon} = 0.$$

Proof. By definition of the q_ε-norm,

$$\begin{aligned}\|\chi_\varepsilon - 1\|_{q_\varepsilon}^2 &= \|\chi_\varepsilon - 1\|_0^2 + \|\nabla(\chi_\varepsilon - 1)\|_0^2 \\ &= \|\chi_\varepsilon - 1\|_0^2 + \|\nabla\chi_\varepsilon\|_0^2.\end{aligned}$$

Since $\chi_\varepsilon - 1$ is bounded and tends to 0 as $\varepsilon \to 0$, the first term in this sum goes to 0 as $\varepsilon \to 0$. If we can show that the limit of the second term is 0, we will be done.

The derivative $\chi_\varepsilon'(r)$ is 0 for $r > \sqrt{\varepsilon}$, so we have

$$\|\nabla\chi_\varepsilon\|_0^2 = \int_{B(p_1,\sqrt{\varepsilon})} |\nabla\chi_\varepsilon|^2 \, dv_{g_1}.$$

Since we are concerned with the limit as ε tends to 0, we may assume that $B(p_1, \sqrt{\varepsilon})$ is contained in an orbifold chart centered at p_1. We are also assuming the metric g_1 is Euclidean near p_1. Thus our chart can be taken to be the quotient of the ball of radius $\sqrt{\varepsilon}$ centered at the origin in $\mathbb{R}^n$ by a finite orthogonal group G. In this case, working in radial coordinates $(r, \theta_2, \ldots, \theta_n)$ about the origin,

$$\begin{aligned}\int_{B(p_1,\sqrt{\varepsilon})} |\nabla\chi_\varepsilon|^2 \, dv_{g_1} &= \int_{B(p_1,\sqrt{\varepsilon})} |d\chi_\varepsilon|^2 \, dv_{g_1} \\ &= \frac{1}{|G|} \int_{B(0,\sqrt{\varepsilon})} \Big(\frac{\partial\chi_\varepsilon}{\partial r}\Big)^2 dr \wedge \star dr \\ &= \frac{1}{|G|} \int_{B(0,\sqrt{\varepsilon})} \Big(\frac{\partial\chi_\varepsilon}{\partial r}\Big)^2 r^{n-1} \, dr \, d\theta_2 \cdots d\theta_n \\ &= \frac{1}{|G|} \int_{B(0,\sqrt{\varepsilon})} \frac{4}{(\ln\varepsilon)^2 r^2} r^{n-1} \, dr \, d\theta_2 \cdots d\theta_n \\ &= \frac{1}{|G|} \frac{4}{(\ln\varepsilon)^2} \int_{S^{n-1}(1)} d\theta_2 \cdots d\theta_n \int_\varepsilon^{\sqrt{\varepsilon}} r^{n-3} \, dr \\ &= \frac{4\mathrm{Vol}(S^{n-1}(1))}{|G|(\ln\varepsilon)^2} \int_\varepsilon^{\sqrt{\varepsilon}} r^{n-3} \, dr.\end{aligned}$$

From here, direct computation in the cases of $n = 2$ and $n > 2$ shows that $\lim_{\varepsilon\to 0} \|\nabla\chi_\varepsilon\|_0^2 = 0$. Thus we have completed our proof. □

Now, before we begin the proof of Proposition 3.1, we state the min-max principle as it applies to our particular situation. We note that by making use of Proposition 2.14 and Theorem 2.15, the standard proof of the min-max principle carries over directly (see, for example, [21, Theorem 4.3.13, p. 302]).

Theorem 3.6 (Min-max principle). *For $f \in H^1(O, g_\varepsilon)$, let*

$$R(f) = \frac{\|\nabla f\|_{L^2(O,g_\varepsilon)}^2}{\|f\|_{L^2(O,g_\varepsilon)}^2}.$$

Then

$$\lambda_k(O, g_\varepsilon) = \inf_{\substack{Y \subset H^1(O,g_\varepsilon) \\ \dim Y = k+1}} \sup_{0 \neq f \in Y} R(f).$$

Using the min-max principle, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\{f_0, f_1, \ldots, f_k\}$ be an orthonormal basis of eigenfunctions of the first $k+1$ eigenvalues on (O_1, g_1), counted with multiplicity. Denote by E the span of $\{f_0, f_1, \dots, f_k\}$. Let $\chi_\varepsilon : O_1 \to [0,1]$ be defined as above. By Lemma 3.4, for small ε the span E_ε of $\{\chi_\varepsilon f_0, \chi_\varepsilon f_1, \dots, \chi_\varepsilon f_k\}$ is a $(k+1)$-dimensional subspace of functions on $(O_1(\varepsilon), g_1)$. By our definition of $H^1(O, g_\varepsilon)$, we can consider each $\chi_\varepsilon f \in E_\varepsilon$ an element of $H^1(O, g_\varepsilon)$ via the extension $\chi_\varepsilon f \mapsto (\chi_\varepsilon f, 0)$. In this way, we now consider E_ε to be a $(k+1)$-dimensional subspace of $H^1(O, g_\varepsilon)$.

Note that every function $u_\varepsilon \in E_\varepsilon$ is of the form $u_\varepsilon = (\chi_\varepsilon f, 0)$ for some function $f \in E$, but the functions $\chi_\varepsilon f_i$ are not necessarily eigenfunctions of the Laplacian acting on (O, g_ε). By the min-max principle,

$$\lambda_k(O, g_\varepsilon) \le \sup_{(0,0) \neq (\chi_\varepsilon f, 0) \in E_\varepsilon} \frac{\|\nabla(\chi_\varepsilon f, 0)\|^2_{L^2(O,g_\varepsilon)}}{\|(\chi_\varepsilon f, 0)\|^2_{L^2(O,g_\varepsilon)}} = \sup_{0 \neq f \in E} \frac{\|\nabla \chi_\varepsilon f\|_0^2}{\|\chi_\varepsilon f\|_0^2}.$$

We now consider the terms in this supremum in more detail, starting from the q_ε-norm.

Observe that by the triangle inequality applied to the q_ε-norm, we have

$$\|\chi_\varepsilon f\|_{q_\varepsilon} \le \|f\|_{q_\varepsilon} + \|\chi_\varepsilon f - f\|_{q_\varepsilon}$$

so

$$\|\chi_\varepsilon f\|^2_{q_\varepsilon} \le \|f\|^2_{q_\varepsilon} + 2\|f\|_{q_\varepsilon}\|\chi_\varepsilon f - f\|_{q_\varepsilon} + \|\chi_\varepsilon f - f\|^2_{q_\varepsilon}. \tag{3.1}$$

Since for any nonzero $f \in E$ and for ε small, $\|\chi_\varepsilon f\|_0^2 \neq 0$, we can say

$$\begin{aligned}\frac{\|\chi_\varepsilon f\|^2_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} &\le \frac{\|f\|^2_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{2\|f\|_{q_\varepsilon}\|\chi_\varepsilon f - f\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{\|\chi_\varepsilon f - f\|^2_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} \\ &= \frac{\|f\|^2_{q_\varepsilon}}{\|f\|_0^2} + \delta^f(\varepsilon)\end{aligned}$$

where

$$\delta^f(\varepsilon) = \frac{\|f\|^2_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} - \frac{\|f\|^2_{q_\varepsilon}}{\|f\|_0^2} + \frac{2\|f\|_{q_\varepsilon}\|\chi_\varepsilon f - f\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{\|\chi_\varepsilon f - f\|^2_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2}.$$

By the definition of the q_ε-norm, for any function f, $\|f\|^2_{q_\varepsilon} = \|f\|_0^2 + \|\nabla f\|_0^2$. Thus

$$\frac{\|\chi_\varepsilon f\|_0^2}{\|\chi_\varepsilon f\|_0^2} + \frac{\|\nabla(\chi_\varepsilon f)\|_0^2}{\|\chi_\varepsilon f\|_0^2} \le \frac{\|f\|_0^2}{\|f\|_0^2} + \frac{\|\nabla f\|_0^2}{\|f\|_0^2} + \delta^f(\varepsilon),$$

and we conclude that

$$\frac{\|\nabla(\chi_\varepsilon f)\|_0^2}{\|\chi_\varepsilon f\|_0^2} \le \frac{\|\nabla f\|_0^2}{\|f\|_0^2} + \delta^f(\varepsilon).$$

Therefore,

$$\begin{aligned}\lambda_k(O, g_\varepsilon) &\le \sup_{0 \neq f \in E} \frac{\|\nabla(\chi_\varepsilon f)\|_0^2}{\|\chi_\varepsilon f\|_0^2} \\ &\le \sup_{0 \neq f \in E} \left[\frac{\|\nabla f\|_0^2}{\|f\|_0^2} + \delta^f(\varepsilon)\right] \\ &\le \sup_{0 \neq f \in E} \left[\frac{\|\nabla f\|_0^2}{\|f\|_0^2}\right] + \sup_{0 \neq f \in E}\left[\delta^f(\varepsilon)\right] \\ &= \lambda_k(O_1, g_1) + \sup_{0 \neq f \in E}\left[\delta^f(\varepsilon)\right],\end{aligned}$$

where the last equality follows from an application of the min-max principle to (O_1, g_1) and the fact that $f_k \in E$.

Now we will produce an upper bound $\delta(\varepsilon)$ for the set $\{\delta^f(\varepsilon) \,|\, 0 \neq f \in E\}$ and show that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. As an initial simplifying step, note that for any nonzero $f \in E$ and nonzero $c \in \mathbb{R}$, by the definition of $\delta^f(\varepsilon)$, $\delta^f(\varepsilon) = \delta^{cf}(\varepsilon)$. Therefore,

$$\sup_{0 \neq f \in E} \left[\delta^f(\varepsilon)\right] = \sup_{f \in E, \|f\|_0 = 1} \left[\delta^f(\varepsilon)\right].$$

We will assume in what follows that $\|f\|_0 = 1$.

Suppose that $f = c_0 f_0 + c_1 f_1 + \cdots + c_k f_k \in E$. We note that since $\|f\|_0 = 1$, $c_0^2 + c_1^2 + \cdots + c_k^2 = 1$. Since $\{f_0, f_1, \ldots, f_k\}$ is an orthonormal basis of eigenfunctions, by Proposition 2.14

$$\begin{aligned}
\|f\|_q^2 &= \|f\|_0^2 + \|\nabla f\|_0^2 \\
&= \|f\|_0^2 + \langle \Delta f, f \rangle_0 \\
&= \|f\|_0^2 + c_0^2 \lambda_0 + c_1^2 \lambda_1 + \cdots + c_k^2 \lambda_k \\
&= 1 + c_0^2 \lambda_0 + c_1^2 \lambda_1 + \cdots + c_k^2 \lambda_k \\
&\leq 1 + (c_0^2 + c_1^2 + \cdots + c_k^2) \lambda_k \\
&= 1 + \lambda_k.
\end{aligned}$$

where $\lambda_i = \lambda_i(O_1, g_1)$. We note that this conclusion follows from the min-max principle for finite-dimensional vector spaces as well, but this is the direct argument.

Thus, using the Cauchy-Schwarz inequality applied to the q_ε-norm we have:

$$\begin{aligned}
\delta^f(\varepsilon) &= \frac{\|f\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2} - \frac{\|f\|_{q_\varepsilon}^2}{\|f\|_0^2} + \frac{2\|f\|_{q_\varepsilon} \|\chi_\varepsilon f - f\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{\|\chi_\varepsilon f - f\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2} \\
&= \frac{\|f\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2} - \|f\|_{q_\varepsilon}^2 + \frac{2\|f\|_{q_\varepsilon} \|\chi_\varepsilon f - f\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{\|\chi_\varepsilon f - f\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2} \\
&= \frac{\|f\|_{q_\varepsilon}^2 - \|f\|_{q_\varepsilon}^2 \|\chi_\varepsilon f\|_0^2}{\|\chi_\varepsilon f\|_0^2} + \frac{2\|f\|_{q_\varepsilon} \|\chi_\varepsilon f - f\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{\|\chi_\varepsilon f - f\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2} \\
&\leq \frac{\|f\|_{q_\varepsilon}^2 (1 - \|\chi_\varepsilon f\|_0^2)}{\|\chi_\varepsilon f\|_0^2} + \frac{2\|f\|_{q_\varepsilon}^2 \|\chi_\varepsilon - 1\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{\|f\|_{q_\varepsilon}^2 \|\chi_\varepsilon - 1\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2} \\
&\leq \frac{(1 + \lambda_k)(1 - \|\chi_\varepsilon f\|_0^2)}{\|\chi_\varepsilon f\|_0^2} + \frac{2(1 + \lambda_k)\|\chi_\varepsilon - 1\|_{q_\varepsilon}}{\|\chi_\varepsilon f\|_0^2} + \frac{(1 + \lambda_k)\|\chi_\varepsilon - 1\|_{q_\varepsilon}^2}{\|\chi_\varepsilon f\|_0^2}.
\end{aligned}$$

By the definitions of χ_ε and the L^2-norm on (O_1, g_1), for any nonzero smooth function f on (O_1, g_1),

$$\|\chi_\varepsilon f\|_0^2 \leq \|f\|_0^2.$$

Furthermore, since $\chi_\varepsilon f \to f$ pointwise as $\varepsilon \to 0$, $\|\chi_\varepsilon f\|_0 \to \|f\|_0$. This holds because O_1 is compact, and therefore pointwise convergence implies uniform convergence. Therefore, there exists a value ε_1 such that for all $\varepsilon < \varepsilon_1$,

$$\|\chi_\varepsilon f\|_0^2 \geq \frac{\|f\|_0^2}{2} = \frac{1}{2}.$$

Thus for $\varepsilon < \varepsilon_1$, we have

$$\begin{aligned}\delta^f(\varepsilon) &\le 2(1+\lambda_k)(1-\|\chi_\varepsilon f\|_0^2) + 4(1+\lambda_k)\|\chi_\varepsilon - 1\|_{q_\varepsilon} + 2(1+\lambda_k)\|\chi_\varepsilon - 1\|_{q_\varepsilon}^2 \\ &\le 2(1+\lambda_k)((1-\|\chi_\varepsilon f\|_0^2) + 2\|\chi_\varepsilon - 1\|_{q_\varepsilon} + \|\chi_\varepsilon - 1\|_{q_\varepsilon}^2).\end{aligned}$$

Now recalling that we denote the volume form on (O_1, g_1) by dv_{g_1},

$$\begin{aligned}1-\|\chi_\varepsilon f\|_0^2 &= \|f\|_0^2 - \|\chi_\varepsilon f\|_0^2 \\ &= \int f^2\, dv_{g_1} - \int \chi_\varepsilon^2 f^2\, dv_{g_1} \\ &= \int (f^2 - f^2\chi_\varepsilon^2)\, dv_{g_1} \\ &= \int f^2(1-\chi_\varepsilon^2)\, dv_{g_1} \\ &= \langle f^2, (1-\chi_\varepsilon^2)\rangle_0 \\ &\le \|f^2\|_0 \|1-\chi_\varepsilon^2\|_0 \\ &\le \|f\|_0^2 \|1-\chi_\varepsilon^2\|_0 \\ &= \|1-\chi_\varepsilon^2\|_0,\end{aligned}$$

by the Cauchy-Schwarz inequality applied to the L_2-norm on (O_1, g_1).

Thus

$$\delta^f(\varepsilon) \le 2(1+\lambda_k)(\|1-\chi_\varepsilon^2\|_0 + 2\|\chi_\varepsilon - 1\|_{q_\varepsilon} + \|\chi_\varepsilon - 1\|_{q_\varepsilon}^2).$$

Letting $\delta(\varepsilon) = 2(1+\lambda_k)(\|1-\chi_\varepsilon^2\|_0) + 2\|\chi_\varepsilon - 1\|_{q_\varepsilon} + \|\chi_\varepsilon - 1\|_{q_\varepsilon}^2)$ and making use of Lemma 3.5, we conclude that

$$\lambda_k(O, g_\varepsilon) = \lambda_k(O_1, g_1) + \delta(\varepsilon)$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore

$$\limsup_{\varepsilon\to 0} \lambda_k(O, g_\varepsilon) = \lambda_k(O_1, g_1)$$

as desired. □

4. A lower bound for $\lambda_k(O, g_\varepsilon)$ as $\varepsilon \to 0$

In this section we will prove Proposition 4.1, which constitutes the second half of Theorem 1.1.

Proposition 4.1. *Suppose that (O_1, g_1) and (O, g_ε) are as defined in Section 1.1. Then*

$$\lambda_k(O_1, g_1) \le \liminf_{\varepsilon\to 0} \lambda_k(O, g_\varepsilon).$$

The proof of Proposition 4.1 will follow directly from Lemmas 4.2 and 4.3 below.

Lemma 4.2. *For each $j = 0, 1, \dots, k$, $\liminf_{\varepsilon\to 0} \lambda_j(O, g_\varepsilon)$ is an eigenvalue of the Laplacian acting on functions on (O_1, g_1).*

Proof. For notational simplicity, throughout the proof let $\alpha_j = \liminf_{\varepsilon\to 0} \lambda_j(O, g_\varepsilon)$.

In order to pass from information about eigenvalues of (O, g_ε) to information about eigenvalues of (O_1, g_1), we must first pass from information about associated eigenfunctions on (O, g_ε) to information about associated eigenfunctions on (O_1, g_1). With this in mind, let $f_{j,\varepsilon} = (f_{j,\varepsilon}^1, f_{j,\varepsilon}^2)$, $j = 0, 1, \dots, k$ be an orthonormal collection of eigenfunctions associated to the first $k+1$ eigenvalues of Δ_ε on (O, g_ε). For each $j = 0, 1, \ldots, k$, consider $f^1_{j,\varepsilon} \in H^1(O_1(\varepsilon), g_1)$. We note that for any j, it is not necessarily the case that $f^1_{j,\varepsilon}$ is an eigenfunction of the Laplacian on $(O_1(\varepsilon), g_1)$ under any choice of boundary conditions.

Since we are assuming that ε is small enough that $B(p_1, \varepsilon)$ is contained in a single chart, for each choice of $\varepsilon > 0$ we can harmonically extend $f^1_{j,\varepsilon}$ to an element $\bar{f}^1_{j,\varepsilon}$ of $H^1(O_1, g_1)$. (In the case that the chart is a quotient, since the Laplacian is invariant under the action of the group, we can lift to the cover, extend harmonically, and then average if necessary.) Furthermore, with this extension, we are guaranteed that there is some constant C, independent of function and indices, such that

$$\|\bar{f}^1_{j,\varepsilon}\|_{H^1(O_1,g_1)} \leq C\|f^1_{j,\varepsilon}\|_{H^1(O_1(\varepsilon),g_1)}$$

(see [22, Example 1, p.40]).

Consider a sequence (ε_i) with $\varepsilon_i \to 0$ as $i \to \infty$. For a particular choice of j, since $\alpha_j = \liminf_{\varepsilon\to 0} \lambda_j(O, g_\varepsilon)$, we may assume that (ε_i) is such that $\alpha_j = \lim_{\varepsilon_i \to 0} \lambda_j(O, g_{\varepsilon_i})$. For this sequence (ε_i), consider the associated sequence $(\bar{f}^1_{j,\varepsilon_i})$. We observe that $(\bar{f}^1_{j,\varepsilon_i})$ is a bounded sequence in $H^1(O_1, g_1)$. Indeed, for each ε_i, since f_{j,ε_i} is part of an orthonormal collection of eigenfunctions of Δ_{ε_i} on (O, g_{ε_i}) we have

$$\begin{aligned}
\|\bar{f}^1_{j,\varepsilon_i}\|^2_{H^1(O_1,g_1)} &\leq C\|f^1_{j,\varepsilon_i}\|^2_{H^1(O_1(\varepsilon_i),g_1)} \\
&\leq C(\|f^1_{j,\varepsilon_i}\|^2_{H^1(O_1(\varepsilon_i),g_1)} + \|f^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),\varepsilon_i^2 g_2)}) \\
&= C(\|f^1_{j,\varepsilon_i}\|^2_{L^2(O_1(\varepsilon_i),g_1)} + \|\nabla f^1_{j,\varepsilon_i}\|^2_{L^2(O_1(\varepsilon_i),g_1)} \\
&\qquad + \|f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)} + \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)}) \\
&= C(\|f_{j,\varepsilon_i}\|^2_{L^2(O,g_{\varepsilon_i})} + q_{\varepsilon_i}(f_{j,\varepsilon_i}, f_{j,\varepsilon_i})) \\
&= C(\|f_{j,\varepsilon_i}\|^2_{L^2(O,g_{\varepsilon_i})} + \lambda_j(O, g_{\varepsilon_i})\|f_{j,\varepsilon_i}\|^2_{L^2(O,g_{\varepsilon_i})}) \\
&= C(1 + \lambda_j(O, g_{\varepsilon_i})) \\
&= C(1 + \lambda_j(O_1, g_1) + \delta(\varepsilon_i)),
\end{aligned}$$

where $\delta(\varepsilon_i) \to 0$ as $\varepsilon_i \to 0$, as proven at the end of Section 3.

Since $(\bar{f}^1_{j,\varepsilon_i})$ is a bounded sequence in $H^1(O_1, g_1)$, there is a subsequence that converges weakly in $H^1(O_1, g_1)$ (see [15, p.639]). Thus, after reindexing, we have a sequence $(\bar{f}^1_{j,\varepsilon_i})$ that converges weakly to an element $\bar{f}^1_j$ in $H^1(O_1, g_1)$ and such that $\lambda_j(O, g_{\varepsilon_i})$ converges to α_j as $\varepsilon_i \to 0$. Furthermore, by Rellich's theorem (Theorem 2.7), the inclusion of $H^1(O_1, g_1)$ in $L^2(O_1, g_1)$ is compact. Therefore, $(\bar{f}^1_{j,\varepsilon_i})$ converges strongly to $\bar{f}^1_j$ in $L^2(O_1, g_1)$.

We now show that for each $j = 0, 1, \ldots, k$, $\bar{f}^1_j$ is an eigenfunction of the Laplacian acting on (O_1, g_1), with eigenvalue α_j. To begin, let ψ be an element in $\mathcal{C}^\infty_0(O_1 \backslash \{p_1\})$, the set of compactly supported smooth functions on $O_1 \backslash \{p_1\}$. This means that ψ is 0 in a neighborhood about p_1 and thus is an element of $H^1(O_1, g_1)$.

We note that $\langle \nabla \bar{f}^1_j, \nabla\psi\rangle_{L^2(O_1,g_1)} = \lim_{\varepsilon_i \to 0}\langle \nabla \bar{f}^1_{j,\varepsilon_i}, \nabla\psi\rangle_{L^2(O_1,g_1)}$ as follows. By definition,

$$\langle \bar{f}^1_j, \psi\rangle_{H^1(O_1,g_1)} = \langle \bar{f}^1_j, \psi\rangle_{L^2(O_1,g_1)} + \langle \nabla \bar{f}^1_j, \nabla\psi\rangle_{L^2(O_1,g_1)}.$$

On the other hand, from the fact that $(\bar{f}^1_{j,\varepsilon_i})$ converges weakly to $\bar{f}^1_j$ in $H^1(O_1, g_1)$,

$$\begin{aligned}\langle \bar{f}^1_j, \psi\rangle_{H^1(O_1,g_1)} &= \lim_{\varepsilon_i\to 0}\langle \bar{f}^1_{j,\varepsilon_i}, \psi\rangle_{H^1(O_1,g_1)}\\ &= \lim_{\varepsilon_i\to 0}\big(\langle \bar{f}^1_{j,\varepsilon_i}, \psi\rangle_{L^2(O_1,g_1)} + \langle \nabla\bar{f}^1_{j,\varepsilon_i}, \nabla\psi\rangle_{L^2(O_1,g_1)}\big).\end{aligned}$$

Since $(\bar{f}^1_{j,\varepsilon_i})$ converges strongly to $\bar{f}^1_j$ in $L^2(O_1, g_1)$,

$$\langle \bar{f}^1_j, \psi\rangle_{L^2(O_1,g_1)} = \lim_{\varepsilon_i\to 0}\langle \bar{f}^1_{j,\varepsilon_i}, \psi\rangle_{L^2(O_1,g_1)}.$$

Thus we conclude that $\langle \nabla\bar{f}^1_j, \nabla\psi\rangle_{L^2(O_1,g_1)} = \lim_{\varepsilon_i\to 0}\langle \nabla\bar{f}^1_{j,\varepsilon_i}, \nabla\psi\rangle_{L^2(O_1,g_1)}$.

We now use the fact that $f_{j,\varepsilon_i} = (f^1_{j,\varepsilon_i}, f^2_{j,\varepsilon_i})$ is an eigenfunction of the Laplacian on (O, g_ε) to compute $\langle \nabla\bar{f}^1_j, \nabla\psi\rangle_{L^2(O_1,g_1)}$ more explicitly. Recall that ψ is 0 on a neighborhood about p_i, so for small enough ε, $\langle \bar{f}^1_{j,\varepsilon_i}, \psi\rangle_{L^2(O_1,g_1)} = \langle f^1_{j,\varepsilon_i}, \psi\rangle_{L^2(O_1(\varepsilon_i),g_1)}$ and $\langle \nabla\bar{f}^1_{j,\varepsilon}, \nabla\psi\rangle_{L^2(O_1,g_1)} = \langle \nabla f^1_{j,\varepsilon}, \nabla\psi\rangle_{L^2(O_1(\varepsilon),g_1)}$. Therefore,

$$\begin{aligned}\langle \nabla\bar{f}^1_j, \nabla\psi\rangle_{L^2(O_1,g_1)} &= \lim_{\varepsilon_i\to 0}\langle \nabla\bar{f}^1_{j,\varepsilon_i}, \nabla\psi\rangle_{L^2(O_1,g_1)}\\ &= \lim_{\varepsilon_i\to 0}\langle \nabla f^1_{j,\varepsilon_i}, \nabla\psi\rangle_{L^2(O_1(\varepsilon_i),g_1)}\\ &= \lim_{\varepsilon_i\to 0}\big(\langle \nabla f^1_{j,\varepsilon_i}, \nabla\psi\rangle_{L^2(O_1(\varepsilon_i),g_1)} + \langle \nabla f^2_{j,\varepsilon_i}, 0\rangle_{L^2(O_2(1),\varepsilon_i^2 g_2)}\big)\\ &= \lim_{\varepsilon_i\to 0} q_{\varepsilon_i}((f^1_{j,\varepsilon_i}, f^2_{j,\varepsilon_i})(\psi, 0))\\ &= \lim_{\varepsilon_i\to 0}\lambda_j(O, g_{\varepsilon_i})\langle (f^1_{j,\varepsilon_i}, f^2_{j,\varepsilon_i}), (\psi, 0)\rangle_{L^2(O,g_\varepsilon)}\\ &= \lim_{\varepsilon_i\to 0}\lambda_j(O, g_{\varepsilon_i})\big(\langle f^1_{j,\varepsilon_i}, \psi\rangle_{L^2(O_1(\varepsilon_i),g_1)} + \langle f^2_{j,\varepsilon_i}, 0\rangle_{L^2(O_2(1),\varepsilon_i^2 g_2)}\big)\\ &= \lim_{\varepsilon_i\to 0}\lambda_j(O, g_{\varepsilon_i})\langle \bar{f}_{j,\varepsilon_i}, \psi\rangle_{L^2(O_1,g_1)}\\ &= \alpha_j\langle \bar{f}^1_j, \psi\rangle_{L^2(O_1,g_1)}.\end{aligned}$$

By a proof almost identical to that of Lemma 1 in [2], we have that $\mathcal{C}^\infty_0(O_1\backslash\{p_1\})$ is dense in $H^1(O_1, g_1)$. Thus $\langle \nabla\bar{f}^1_j, \nabla\psi\rangle_{L^2(O_1,g_1)} = \alpha_j\langle \bar{f}^1_j, \psi\rangle_{L^2(O_1,g_1)}$ for all $\psi \in H^1(O_1, g_1)$, making $\bar{f}^1_j$ a weak solution for the Laplacian acting on (O_1, g_1). Since the regularity theorem for weak solutions holds for orbifolds (see [10]), we conclude that in fact, $\bar{f}^1_j$ is an element of $\mathcal{C}^\infty(O_1)$ and $\Delta\bar{f}^1_j = \alpha_j\bar{f}^1_j$.

Thus, for every $j = 0, 1, \ldots, k$, α_j is an eigenvalue of the Laplacian acting on (O_1, g_1), as desired. □

We now turn to the second lemma that we will need to complete the proof of Proposition 4.1.

Lemma 4.3. *For each $k \in \mathbb{N}$, $\liminf_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon) = \lambda_l(O_1, g_1)$ for some $l \geq k$.*

Proof. For $j = 0, 1, \ldots, k$, let $\bar{f}^1_j$ be as in the proof of Lemma 4.2. From the proof Lemma 4.2, we know that $\bar{f}^1_j$ is an eigenfunction of the Laplacian acting on (O_1, g_1) with eigenvalue equal to $\liminf_{\varepsilon\to 0}\lambda_j(O, g_\varepsilon)$. Thus, for our particular choice of k, $\liminf_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon) = \lambda_l(O_1, g_1)$ for some l. We will show that $l \geq k$ by showing that $\{\bar{f}^1_0, \bar{f}^1_1, \ldots, \bar{f}^1_k\}$ is an orthonormal set. If this is the case, then these eigenfunctions correspond to distinct eigenvalues in the Laplace spectrum of (O_1, g_1). These eigenvalues may or may not be the first $k + 1$ eigenvalues in the spectrum, but at least we can say that $\liminf_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon) = \lambda_l(O_1, g_1)$ for some $l \geq k$.

Our main tool in showing orthogonality will be to show that the second components, f^2_{j,ε_i}, of the eigenfunctions of the Laplacian acting on (O, g_{ε_i}) tend to 0 as $\varepsilon_i \to 0$. We need a common space of comparison so we begin by considering $\tilde{f}^2_{j,\varepsilon_i} = \varepsilon_i^{n/2} f^2_{j,\varepsilon_i}$ in $H^1(O_2(1), g_2)$. The sequence $(\tilde{f}^2_{j,\varepsilon_i})$ is bounded in $H^1(O_2(1), g_2)$ as follows. By a conformal change of variables, for $\varepsilon_i > 0$,

$$\varepsilon_i^n \|f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} = \|f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)},$$
$$\varepsilon_i^{n-2} \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} = \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)}.$$

This implies that

$$\begin{aligned}
\|\tilde{f}^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),g_2)} &= \|\varepsilon_i^{n/2} f^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),g_2)} \\
&= \varepsilon_i^n \|f^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),g_2)} \\
&= \varepsilon_i^n (\|f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} + \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)}) \\
&= \|f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)} + \varepsilon_i^2 \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)} \\
&\le \|f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)} + \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)} \\
&= \|f^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),\varepsilon_i^2 g_2)}.
\end{aligned}$$

An argument almost identical to the one above that $(\bar{f}^1_{j,\varepsilon_i})$ is a bounded sequence in $H^1(O_1, g_1)$ shows that $(\tilde{f}^2_{j,\varepsilon_i})$ is a bounded sequence in $H^1(O_2(1), g_2)$. Thus it has a subsequence that converges weakly in $H^1(O_2(1), g_2)$ and strongly in $L^2(O_2(1), g_2)$. Call the limit $\tilde{f}^2_j$. We will show that $\tilde{f}^2_j = 0$.

The norm on any Hilbert space is sequentially lower-semicontinuous with respect to the weak topology on the space (see [15, p.639]). Thus, since $(\tilde{f}^2_{j,\varepsilon_i})$ converges weakly to $\tilde{f}^2_j$ in $H^1(O_2(1), g_2)$,

$$\|\tilde{f}^2_j\|^2_{H^1(O_2(1),g_2)} \le \liminf_{\varepsilon_i \to 0} \|\tilde{f}^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),g_2)},$$

i.e.

$$\begin{aligned}
\|\tilde{f}^2_j\|^2_{L^2(O_2(1),g_2)} &+ \|\nabla \tilde{f}^2_j\|^2_{L^2(O_2(1),g_2)} \\
&\le \liminf_{\varepsilon_i \to 0} \big(\|\tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} + \|\nabla \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)}\big).
\end{aligned}$$

But because $(\tilde{f}^2_{j,\varepsilon_i})$ converges strongly to $\tilde{f}^2_j$ in $L^2(O_2(1), g_2)$, we conclude that

$$\|\nabla \tilde{f}^2_j\|^2_{L^2(O_2(1),g_2)} \le \liminf_{\varepsilon_i \to 0} \|\nabla \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)}.$$

Now,

$$\begin{aligned}
\|\nabla \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} &\le \big(\varepsilon_i^2 \|\nabla f^1_{j,\varepsilon_i}\|^2_{L^2(O_1(\varepsilon_i),g_1)} + \|\nabla \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)}\big) \\
&= \big(\varepsilon_i^2 \|\nabla f^1_{j,\varepsilon_i}\|^2_{L^2(O_1(\varepsilon_i),g_1)} + \varepsilon_i^n \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)}\big) \\
&= \varepsilon_i^2 \big(\|\nabla f^1_{j,\varepsilon_i}\|^2_{L^2(O_1(\varepsilon_i),g_1)} + \|\nabla f^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),\varepsilon_i^2 g_2)}\big) \\
&= \varepsilon_i^2 q_{\varepsilon_i}(f_{j,\varepsilon}, f_{j,\varepsilon}) \\
&= \varepsilon_i^2 \lambda_j(O, g_{\varepsilon_i}) \|f_{j,\varepsilon_i}\|_{L^2(O,g_{\varepsilon_1})} \\
&= \varepsilon_i^2 \lambda_j(O, g_{\varepsilon_i}) \\
&= \varepsilon_i^2 (\lambda_j(O_1, g_1) + \delta(\varepsilon_i)),
\end{aligned}$$

where $\delta(\varepsilon_i) \to 0$, as in Section 3, and therefore is bounded. Thus, we have $\|\nabla \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} \to 0$ as $\varepsilon_i \to 0$. This in turn implies that $\nabla \tilde{f}^2_j = 0$ and $\tilde{f}^2_j$ is constant.

Before completing the proof that $\tilde{f}^2_j$ is 0, we observe that the sequence $(\tilde{f}^2_{j,\varepsilon_i})$ in fact converges strongly to $\tilde{f}^2_j$ in $H^1(O_2(1), g_2)$. Indeed,

$$\begin{aligned}
\lim_{\varepsilon_i \to 0} \|\tilde{f}^2_j - &\tilde{f}^2_{j,\varepsilon_i}\|^2_{H^1(O_2(1),g_2)} \\
&= \lim_{\varepsilon_i \to 0} \left(\|\tilde{f}^2_j - \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} + \|\nabla \tilde{f}^2_j - \nabla \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} \right) \\
&= \lim_{\varepsilon_i \to 0} \left(\|\tilde{f}^2_j - \tilde{f}^2_{j,\varepsilon_i}\|^2_{L^2(O_2(1),g_2)} + \|\nabla \tilde{f}^2_{j,\varepsilon_i}\|_{L^2(O_2(1),g_2)} \right) \\
&= 0
\end{aligned}$$

since $(\tilde{f}^2_{j,\varepsilon_i})$ converges strongly to $\tilde{f}^2_j$ in $L^2(O_2(1), g_1)$.

Now, consider the trace of $\tilde{f}^2_j$ on the boundary of $O_2(1)$. By the trace theorem (see Remark 2.9), there is a constant C independent of function and indices such that

$$\begin{aligned}
\|\tilde{f}^2_{j_{|\partial O_2(1)}}&\|_{L^2(\partial O_2(1),\partial g_2)} \\
&\leq \|(\tilde{f}^2_j - \tilde{f}^2_{j,\varepsilon_i})_{|\partial O_2(1)}\|_{L^2(\partial O_2(1),\partial g_2)} + \|\tilde{f}^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial g_2)} \\
&\leq C\|\tilde{f}^2_j - \tilde{f}^2_{j,\varepsilon_i}\|_{H^1(O_2(1),g_2)} + \|\tilde{f}^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial g_2)}.
\end{aligned}$$

If we can show that $\|\tilde{f}^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial g_2)} \to 0$ as $\varepsilon_i \to 0$, then since $\tilde{f}^2_j$ is constant, we can conclude that $\tilde{f}^2_j = 0$ on all of $(O_2(1), g_2)$.

By a change of variables on the $(n-1)$-dimensional boundary $\partial O_2(1)$ and the gluing conditions on $H^1(O, g_{\varepsilon_i})$,

$$\begin{aligned}
\|\tilde{f}^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial g_2)} &= \|\varepsilon_i^{n/2} f^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial g_2)} \\
&= \sqrt{\varepsilon_i \varepsilon_i^{n-1}} \|f^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial g_2)} \\
&= \sqrt{\varepsilon_i} \|f^2_{j,\varepsilon_{i_{|\partial O_2(1)}}}\|_{L^2(\partial O_2(1),\partial \varepsilon_i^2 g_2)} \\
&= \sqrt{\varepsilon_i} \|f^1_{j,\varepsilon_{i_{|\partial O_1(\varepsilon_i)}}}\|_{L^2(\partial O_1(\varepsilon_i),\partial g_1)}.
\end{aligned}$$

The results from [3, p.275] can be applied here to conclude that in our situation, for $x \in \partial O_1(\varepsilon_i)$,

$$|f^1_{j,\varepsilon_i}(x)|^2 \leq \int_\varepsilon^{r_0} \frac{1}{r^{n-1}}\, dr \|f^1_{j,\varepsilon_i}\|^2_{H^1(O_1(\varepsilon_i),g_1)},$$

where r_0 is the maximum distance from p_1 of points in the support of f^1_{j,ε_i}. Direct computation then implies that

$$\sqrt{\varepsilon_i} \|f^1_{j,\varepsilon_{i_{|\partial O_1(\varepsilon_i)}}}\|_{L^2(\partial O_1(\varepsilon_i),\partial g_1)} \leq \begin{cases} C\varepsilon_i \sqrt{|\log \varepsilon_i|} \|f^1_{j,\varepsilon_i}\|_{H^1(O_1(\varepsilon_i),g_1)} & \text{if } n = 2 \\ C\varepsilon_i \|f^1_{j,\varepsilon_i}\|_{H^1(O_1(\varepsilon_i),g_1)} & \text{if } n \geq 3. \end{cases}$$

By the work above, $\|f^1_{j,\varepsilon_i}\|_{H^1(O_1(\varepsilon_i),g_1)}$ is bounded, from which it follows that $\lim_{\varepsilon_i\to 0}\|\tilde{f}^2_{j,\varepsilon_i}{}_{|_{\partial O_2(1)}}\|_{L^2(\partial O_2(1),\partial g_2)} = 0$. Therefore for all $j = 0, 1, \ldots, k$, $\tilde{f}^2_j = 0$.

Finally, we are ready to prove that $\{\bar{f}^1_0, \bar{f}^1_1, \ldots, \bar{f}^1_k\}$ is an orthonormal collection of eigenfunctions in $L^2(O_1, g_1)$. By the strong L^2-convergence of $(\bar{f}^2_{j,\varepsilon_i})$, for any $j, l \in 0, 1, \ldots k$ we have

$$\begin{aligned}\langle \bar{f}^1_j, \bar{f}^1_l\rangle_{L^2(O_1,g_1)} &= \lim_{\varepsilon_i\to 0}\langle \bar{f}^1_{j,\varepsilon_i}, \bar{f}^1_{l,\varepsilon_i}\rangle_{L^2(O_1,g_1)}\\ &= \lim_{\varepsilon_i\to 0}\big(\langle f^1_{j,\varepsilon_i}, f^1_{l,\varepsilon_i}\rangle_{L^2(O_1(\varepsilon_i),g_1)} + \langle \bar{f}^1_{j,\varepsilon_i}, \bar{f}^1_{l,\varepsilon_i}\rangle_{L^2(B(p_1,\varepsilon_i),g_1)}\big).\end{aligned}$$

But

$$\lim_{\varepsilon_i\to 0}\langle \bar{f}^1_{j,\varepsilon_i}, \bar{f}^1_{l,\varepsilon_i}\rangle_{L^2(B(p_1,\varepsilon_i),g_1)} = 0.$$

Therefore,

$$\begin{aligned}\langle \bar{f}^1_j, \bar{f}^1_l\rangle_{L^2(O_1,g_1)} &= \lim_{\varepsilon_i\to 0}\big(\langle f^1_{j,\varepsilon_i}, f^1_{l,\varepsilon_i}\rangle_{L^2(O_1(\varepsilon_i),g_1)}\big)\\ &= \lim_{\varepsilon_i\to 0}\big(\langle f^1_{j,\varepsilon_i}, f^1_{l,\varepsilon_i}\rangle_{L^2(O_1(\varepsilon_i),g_1)} + \langle f^2_{j,\varepsilon_i}, f^2_{l,\varepsilon_i}\rangle_{L^2(O_2(1),\varepsilon_i^2 g_2)}\\ &\qquad - \langle f^2_{j,\varepsilon_i}, f^2_{l,\varepsilon_i}\rangle_{L^2(O_2(1),\varepsilon_i^2 g_2)}\big)\\ &= \lim_{\varepsilon_i\to 0}\big(\langle f_{j,\varepsilon_i}, f_{l,\varepsilon_i}\rangle_{L^2(O,g_{\varepsilon_i})} - \varepsilon_i^n\langle f^2_{j,\varepsilon_i}, f^2_{l,\varepsilon_i}\rangle_{L^2(O_2(1),g_2)}\big)\\ &= \lim_{\varepsilon_i\to 0}\big(\langle f_{j,\varepsilon_i}, f_{l,\varepsilon_i}\rangle_{L^2(O,g_{\varepsilon_i})} - \langle \tilde{f}^2_{j,\varepsilon_i}, \tilde{f}^2_{l,\varepsilon_i}\rangle_{L^2(O_2(1),g_2)}\big)\\ &= \lim_{\varepsilon_i\to 0}\big(\delta_{jl} - \langle \tilde{f}^2_{j,\varepsilon_i}, \tilde{f}^2_{l,\varepsilon_i}\rangle_{L^2(O_2(1),g_2)}\big)\\ &= \delta_{jl} - \langle \tilde{f}^2_j, \tilde{f}^2_l\rangle_{L^2(O_2(1),g_2)}\\ &= \delta_{jl}\end{aligned}$$

since $\tilde{f}^2_j = \tilde{f}^2_l = 0$. □

Proof of Proposition 4.1. By Lemma 4.2, $\liminf_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon)$ is an eigenvalue of the Laplacian acting on functions on (O_1, g_1). Thus, $\liminf_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon) = \lambda_l(O_1, g_1)$ for some l. Lemma 4.3 tells us that $l \geq k$. From this we conclude that $\lambda_k(O_1, g_1) \leq \liminf_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon)$, as desired. □

Finally, the statement of Theorem 1.1, that $\lambda_k(O_1, g_1) = \lim_{\varepsilon\to 0}\lambda_k(O, g_\varepsilon)$, follows directly from Propositions 3.1 and 4.1.

5. Continuity of the eigenvalues and the proof of Theorem 1.2

In this section, we prove Theorem 1.2. Specifically, following [28, Section 5], we explain how the results of [7, Theorem 2.2] extend to closed orbifolds, implying that the eigenvalues $\lambda_k(O, g_\varepsilon)$ vary continuously with respect to the metric. Therefore, we may obtain Theorem 1.2 by approximating the non-smooth metrics g_ε with smooth metrics on O.

Let O be a closed orbifold, and let $Q := \mathrm{S}^2(TO)$ denote the symmetric tensor product of the tangent bundle TO of O with itself. We fix an orbifold atlas for O and define an atlas for Q by defining for each chart $(\tilde{U}, \Gamma_U, \pi_U)$ for O the chart $\tilde{V} := (\mathrm{S}^2(T\tilde{U}), \Gamma_V, \pi_V)$ where the action of Γ_V on $\tilde{V}$ is that induced by the action on $\tilde{U}$; see [1, Definition 1.27]. Following [9, Definition 6], for $1 \leq r \leq \infty$, we let $\mathcal{C}^r_{\star\,\mathrm{Orb}}(O, Q)$ denote the collection of complete $\mathcal{C}^r$ orbifold maps from O to Q. With respect to these fixed atlases, a complete $\mathcal{C}^r$ orbifold map is given by a map of underlying spaces along with, for each chart $(\tilde{U}, \Gamma_U, \pi_U)$ for O, a group homomorphism $\Gamma_U \to \Gamma_V$ and corresponding equivariant $\mathcal{C}^r$-lift $\tilde{U} \to \tilde{V}$. Then by [9, Theorem 1], the space $\mathcal{C}^r_{\star\,\mathrm{Orb}}(O, Q)$ equipped with the $\mathcal{C}^\infty$ topology is a smooth $\mathcal{C}^\infty$ manifold locally modeled on a Fréchet space.

Let $\pi : Q \to O$ denote the projection map, with local homomorphisms $\Gamma_U \to \Gamma_V$ given by the identity homomorphism and local lifts given by the projections $\tilde{\pi} : \tilde{V} \to \tilde{U}$. Let $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ denote the subspace of $\mathcal{C}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ consisting of those $f : O \to Q$ such that $\pi \circ f$ is equal to the (complete orbifold) identity map on O. Then $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ is a closed subset of $\mathcal{C}^\infty_{\star\,\mathrm{Orb}}(O, Q)$, as it is the preimage of the singleton given by the identity map under the smooth map $\mathcal{C}^\infty_{\star\,\mathrm{Orb}}(O, Q) \to \mathcal{C}^\infty_{\star\,\mathrm{Orb}}(O, O)$ given by composition with π. Moreover, the distance function defined in [8, Definition 36] corresponds to the usual Fréchet distance function for manifolds, see [7, Section 1.1], when applied to elements of $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$, implying that this distance function is translation-invariant with respect to the vector space structure on the space of sections $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$. It follows that $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ has the structure of a Fréchet space.

We now explain how the results of [7, Sections 1.2 and 2] extend readily to orbifolds. Fix a point $x \in O$, let P_x denote the set of symmetric positive definite forms on $T_xO \times T_xO$, and let S_x denote the set of symmetric forms on $T_xO \times T_xO$. Note that if $(\tilde{U}, \Gamma_U, \pi_U)$ is an orbifold chart for O such that $\pi_U(0) = x$, then elements of P_x correspond to Γ_U-invariant symmetric positive definite forms on $T_0\tilde{U} \times T_0\tilde{U}$, and similarly for elements of S_x. For $\varphi, \psi \in S_x$, we define $\varphi < \psi$ to mean that $\psi - \varphi$ is positive definite. Bando and Urakawa define the distance ρ''_x on P_x by

$$\rho''_x(\varphi, \psi) = \inf\{\delta > 0 \,|\, \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\}.$$

Then [7, Lemma 1.1] demonstrates that a nonsingular linear transformation $T_xO \to T_xO$ induces an isometry on (P_x, ρ''_x), that (P_x, ρ''_x) forms a complete metric space, and that the application of $\varphi \in P_x$ to an element of $T_xO \times T_xO$ yields a continuous function on P_x with respect to ρ''_x. The proofs of these statements extend directly to the orbifold case by simply restricting to Γ_U-invariant forms and noting that limits of Γ_U-invariant forms with respect to ρ''_x are Γ_U-invariant.

We continue to follow [7] by defining metrics ρ'' and ρ on $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ as

$$\rho''(g_1, g_2) = \sup_{x \in O} \rho''_x((g_1)_x, (g_2)_x)$$

and

$$\rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2),$$

where $g_1, g_2 \in \mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ and $\rho'(g_1, g_2)$ denotes the Fréchet norm of $g_1 - g_2$. Then the proof of [7, Proposition 1.2], that $\big(\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q), \rho\big)$ is a complete metric space, again extends directly to the orbifold case by the above observations.

With this, we have the following extension of [7, Theorem 2.2] to orbifolds.

Theorem 5.1. *Let O be a closed orbifold and $\delta > 0$. If $g_1, g_2 \in \mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ such that $\rho(g_1, g_2) < \delta$, then for each $k = 0, 1, 2 \ldots$,*

$$\exp(-(n+1)\delta) \le \frac{\lambda_k(O, g_1)}{\lambda_k(O, g_2)} \le \exp((n+1)\delta).$$

Therefore, λ_k is a continuous function of $\mathrm{Sec}^\infty_{\star\,\mathrm{Orb}}(O, Q)$ equipped with the $\mathcal{C}^0$ topology.

Proof. Bando and Urakawa prove this result for manifolds by showing that if $\rho''(g_1, g_2) < \delta$ and if f is any smooth function with support contained in a neighborhood U, then

$$\exp(-(n+1)\delta)\frac{\|\nabla f\|_{g_2}^2}{\|f\|_{g_2}^2} \leq \frac{\|\nabla f\|_{g_1}^2}{\|f\|_{g_1}^2} \leq \exp((n+1)\delta)\frac{\|\nabla f\|_{g_2}^2}{\|f\|_{g_2}^2}. \tag{5.1}$$

They then patch the local result (5.1) together using a partition of unity and apply the min-max principle in their setting ([7, Proposition 2.1]) to conclude that

$$\exp(-(n+1)\delta)\lambda_k(O, g_2) \leq \lambda_k(O, g_1) \leq \exp((n+1)\delta)\lambda_k(O, g_2). \tag{5.2}$$

Of course, the local estimate given by Equation (5.1) holds in a chart U when restricted to Γ_U-invariant g_1, g_2, and f. Since orbifolds admit partitions of unity, we can make a similar argument and apply our min-max principle (Theorem 3.6) to conclude that Equation (5.2) holds in our setting as well. Because this argument uses Theorem 3.6, which holds for piecewise smooth metrics, the continuity of λ_k extends to such piecewise smooth metrics in the $\mathcal{C}^0$ topology. □

With this, we now turn to the proof of Theorem 1.2, following [28, Section 5]. Fix k and let $\eta > 0$. By Theorem 1.1, there is an $\varepsilon > 0$ such that for $j = 0, 1, \ldots, k$, we have

$$|\lambda_j(O, g_\varepsilon) - \lambda_j(O_1, g_1)| < \frac{\eta}{2}$$

for the piecewise smooth metric g_ε. Choosing a sequence of smooth metrics g_i that converge to g_ε in the $\mathcal{C}^0$-topology, Theorem 5.1 implies that there is an l such that

$$|\lambda_j(O, g_l) - \lambda_j(O, g_\varepsilon)| < \frac{\eta}{2}.$$

Setting $g_{\eta,k} = g_l$ completes the proof.

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