LINEAR ALGEBRA
EXAM 3
SPRING 2007

66 total

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted.

(4) (1) Analyze the long-term behavior (sometimes called the steady-state response) of the dynamical system defined by $\mathbf{x_{k+1}} = A\mathbf{x_k}$, $(k = 0, 1, 2\ldots)$ where

$A = \begin{bmatrix} .8 & 0 \\ 0 & .64 \end{bmatrix}$

and

$\mathbf{x_0} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$.

A is a triangular matrix, so its eigenvalues are the diagonal entries, .8 and .64. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which are linearly independent.

$\vec{x}_0 = 10\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then $\vec{x}_k = 10(.8)^k\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6(.64)^k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus $\lim_{k\to\infty} \vec{x}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Date: April 19, 2007.

6 (2) Let $W = Span\{v_1, \dots v_p\}$. Show that if $\mathbf{x}$ is orthogonal to each v_j, for $1 \le j \le p$, then $\mathbf{x}$ is orthogonal to every vector in W.

let $\vec{w}$ be an arbitrary vector in W.

Thus $\exists$ weights $c_1, \dots c_p$ s.t.

$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p$.

We must show $\vec{x} \cdot \vec{w} = 0$

$$\vec{x} \cdot \vec{w} = x \cdot (c_1 \vec{v}_1 + \dots + c_p \vec{v}_p)$$

$$= c_1 x \cdot \vec{v}_1 + \dots + c_p x \cdot \vec{v}_p$$

$$= 0 + \dots + 0$$

since each v_j is orth. to x.

$$= 0$$

$\boxtimes$

10 (3) Given the following matrix:

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$$

2 (a) Determine the characteristic polynomial of A.

Compute $\det(A - \lambda I) = \det \begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix}$

$= (7-\lambda)(3-\lambda)+4 = 25 - 10\lambda + \lambda^2$

2 (b) Determine the eigenvalues of A.

Eigenvalues of A correspond to roots of char. poly. Thus,

$(25 - 10\lambda + \lambda^2) = (\lambda - 5)^2 = 0$

$\Rightarrow \lambda = 5$ with multiplicity 2

2 (c) For each of the eigenvalues determine a basis for the corresponding eigenspace.

Consider

$A\vec{x} = 5\vec{x}$

$\Leftrightarrow (A - 5I)\vec{x} = \vec{0}$

$\Leftrightarrow \cancel{\left[\begin{array}{cc|c} 5 & -2 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 9/5 & 0 \end{array}\right]}$

$\sim \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Sol. $x_1 = x_2$, x_2 free $\Rightarrow \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

basis for eigenspace

2 (d) Explain why, or why not, the matrix A is diagonalizable (you need not give the diagonalization if one exists).

The matrix A is not diagonalizable since the dimension of the eigen space is smaller than the multiplicity of the eigenvalue

2 (e) Give a benefit of having a diagonalization of a matrix.

Having a diagonalization of a matrix allows:

- easier computation of the powers of the matrix.

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(4) The following message, 0111000, has been received via transmission over a noisy channel and was encoded using the Hamming(7,4) code (the check matrix H is given below). If at most one error has occurred in transmission, determine if an error has occurred and, if so, correct it.

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Compute to see if message is in Nul H.

2 $$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{h}_5 \neq \vec{0}$$

2 Thus an error in transmission has occurred, in the 5th entry. Thus the message sent was

0111100.

2 (5) If precisely two errors occur in the transmission of a vector $\mathbf{x}$, what can you say?

We can say that a error has occurred, but we would "correct" to the wrong message.

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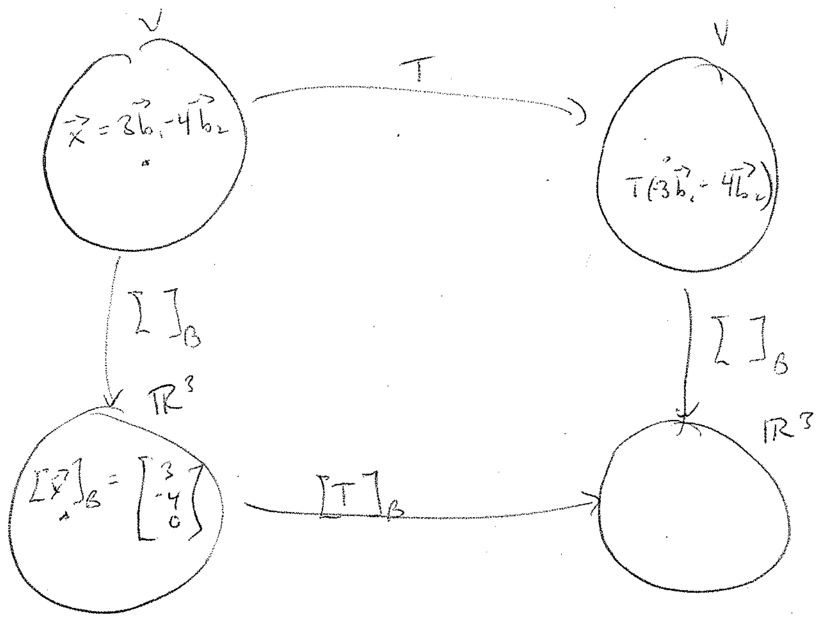
(6) Let $\mathbb{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ be a basis for a vector space V. Find $[T(3\mathbf{b_1} - 4\mathbf{b_2})]_{\mathcal{B}}$? when T is a linear transformation from V to V whose matrix relative to $\mathbb{B}$ is

$$[T]_{\mathbb{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$



$$\begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} +24 \\ -20 \\ 11 \end{bmatrix} =$$

$$[T]_B \, [3b_1 - 4b_2]_B = [T(3b_1 - 4b_2)]_B$$

Thus $T(3b_1 - 4b_2) = 24\vec{b}_1 - 20\vec{b}_2 + 11\vec{b}_3$

12 (7) **True or False:** Justify each answer by citing an appropriate definition or theorem. If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

- 3 If $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

  False.

- 3 If A is invertible, then A is diagonalizable.

  False

- 3 The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

  True.

3 (8) Define: **orthonormal set**.

A set of vectors which are pairwise orthogonal and each have unit length

(6) (9) Let W be the subspace spanned by the $\mathbf{u}$'s.

2 (a) Find the closest point, $\hat{\mathbf{y}}$, to $\mathbf{y}$ in the subspace W.

$$\mathbf{y} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} \quad \mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{u_2} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

$$\hat{y} = \text{proj}_W \vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

$$= \frac{\begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}}{\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}}\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

$$= \frac{12}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}$$

2 (b) Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\vec{y} = \hat{y} + \vec{z} \quad \Leftrightarrow \quad \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} + \vec{z} \quad \Rightarrow \quad \vec{z} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{y} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

2 (c) Find the distance from $\mathbf{y}$ to W.

distance equals $\|\vec{y} - \hat{y}\| = \|\vec{z}\|$

$$= \left\|\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}\right\|$$

$$= \sqrt{36}$$

$$= 6$$

(6) (10) The given set is a basis for a subspace W^*.

2 (a) Apply the Gram-Schmidt process to this basis to produce an orthogonal one.

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{x_2} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \quad \mathbf{x_3} = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{(-36)}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{v}_1 \cdot x_3}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

2 (b) Now produce an orthonormal basis for W^*.

$$\|\vec{v}_1\| = \sqrt{12} \qquad \vec{u}_1 = \begin{bmatrix} -1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix}$$

$$\|\vec{v}_2\| = \sqrt{12}$$

2 (c) Show how to find a QR factorization of $A = [\mathbf{x_1}\ \mathbf{x_2}\ \cancel{\mathbf{x_3}}]$, you need not complete the calculation.

$$Q = [\vec{u}_1\ \vec{u}_2]$$

If $A = QR$, then

$$Q^T A = Q^T Q R$$

$$Q^T A = R$$

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(11) Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

The columns of A are orthogonal so we can compute $\hat{x}$ by finding the projection of $\vec{b}$ onto Col A.

$$\hat{b} = \frac{\vec{x}_1 \cdot \vec{b}}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1 + \frac{\vec{x}_2 \cdot \vec{b}}{\vec{x}_2 \cdot \vec{x}_2} \vec{x}_2 + \frac{\vec{x}_3 \cdot \vec{b}}{\vec{x}_3 \cdot \vec{x}_3} \vec{x}_3$$

$$= \frac{1}{3} \vec{x}_1 + \frac{14}{3} \vec{x}_2 + \frac{-5}{3} \vec{x}_3$$

(4) Thus $\hat{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$

Compute the least-squares error associated with this solution.

$$\| \vec{b} - A\hat{x} \| = \left\| \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -3 \\ 3 \\ 3 \\ 0 \end{bmatrix} \right\| = \sqrt{27} = 3\sqrt{3}$$

(2)

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(12) Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.