LINEAR ALGEBRA
EXAM 3
SPRING 2007

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. Best of luck.

(1) Analyze the long-term behavior (sometimes called the steady-state response) of the dynamical system defined by $\mathbf{x_{k+1}} = A\mathbf{x_k}$, $(k = 0, 1, 2 \ldots)$ where
$A = \begin{bmatrix} .8 & 0 \\ 0 & .64 \end{bmatrix}$
and
$\mathbf{x_0} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$.

Date: April 19, 2007.

(2) Let $W = Span\{v_1, \dots v_p\}$. Show that if $\mathbf{x}$ is orthogonal to each v_j, for $1 \leq j \leq p$, then $\mathbf{x}$ is orthogonal to every vector in W.

(3) Given the following matrix:

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

(b) Determine the eigenvalues of A.

(c) For each of the eigenvalues determine a basis for the corresponding eigenspace.

(d) Explain why, or why not, the matrix A is diagonalizable (you need not give the diagonalization if one exists).

(e) Give a benefit of having a diagonalization of a matrix.

(4) The following message, 0111000, has been received via transmission over a noisy channel and was encoded using the Hamming(7,4) code (the check matrix H is given below). If at most one error has occurred in transmission, determine if an error has occurred and, if so, correct it.

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(5) If precisely two errors occur in the transmission of a vector $\mathbf{x}$, what can you say?

(6) Let $\mathbb{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ be a basis for a vector space V. Find $T(3\mathbf{b_1} - 4\mathbf{b_2})$ when T is a linear transformation from V to V whose matrix relative to $\mathbb{B}$ is

$$[T]_{\mathbb{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

(7) **True or False:** Justify each answer by citing an appropriate definition or theorem. If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

- If $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
- If A is invertible, then A is diagonalizable.
- The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

(8) Define: **orthonormal set**.

(9) Let W be the subspace spanned by the $\mathbf{u}$'s.

(a) Find the closest point, $\hat{\mathbf{y}}$, to $\mathbf{y}$ in the subspace W.

$$\mathbf{y} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u_2} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

(b) Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

(c) Find the distance from $\mathbf{y}$ to W.

(10) The given set is a basis for a subspace W^*.

(a) Apply the Gram-Schmidt process to this basis to produce an orthogonal one.

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x_2} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$$

(b) Now produce an orthonormal basis for W^*.

(c) Show how to find a QR factorization of $A = [\mathbf{x_1}\ \mathbf{x_2}]$, you need not complete the calculation.

(11) Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

$$A= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

Compute the least-squares error associated with this solution.

(12) Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.