Linear Algebra
Exam 2
Spring 2007

April 19, 2007

Total 70

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted.

See page 238

4

1. **Define: basis.**

A set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is said to be a basis of a vector space V if each $\vec{x} \in V$ can be written as a linear combination of the elements of S and further S is linearly independent.

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2. **Define: column space.**

The column space of an $m \times n$ matrix A is the set of all linear combinations of the columns of A; it is a subspace of $\mathbb{R}^m$.

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3. Find an LU factorization of the following matrix.

$$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} = U$$

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$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}$$

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section 2.5

page 149 no. 10

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4. In what context is an LU factorization useful?

See discussion on bottom of page 142, and numerical note on page 146.

If we have a sequence of equations all involving the same coefficient matrix, then using an LU factorization of A to solve these equations will be more efficient.

Atsos,

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5. Compute the determinant of the following matrix. The fewer steps you make in the computation the more points you will be awarded (but please do show your work).

$$G = \begin{bmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{bmatrix}$$

Expand down the ~~first~~ 2nd column

$$\det G = (-1)^{1+2} \cdot 1 \cdot \det \begin{bmatrix} 9 & 9 & 9 & 2 \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 7 & 0 \end{bmatrix} = -1 \cdot \det G_1$$

Expand down 4th Col.

$$\det G_1 = (-1)^{1+4} \cdot 2 \cdot \det \begin{bmatrix} 4 & 0 & 5 \\ 9 & 3 & 9 \\ 6 & 0 & 7 \end{bmatrix} = -2 \cdot \det G_2$$

Expand down 2nd col.

$$\det G_2 = (-1)^{2+2} \cdot 3 \cdot \det \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = 3 \cdot (-2) = -6$$

So $\det G = (-1) \cdot (-2)(-6) = -12$

ch 3, supplementary problems, answer -12

6. **True or False:** Justify each answer by citing an appropriate definition or theorem. If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

3 each

- If one row of a square matrix A is multiplied by k to produce matrix B, then $\det B = \frac{1}{k} detA$.

  False, should be $kdetA$.

- $\mathbb{R}^3$ is a subspace of $\mathbb{R}^4$.

  False, there exists a subspace isomorphic to $\mathbb{R}^3$.

- If B is an echelon form of a matrix A, then the pivot columns of B form a basis for $ColA$.

  False, it is the pivot columns of A that form a basis for $ColA$.

- Since the coordinate mapping is one-to-one, if a set of vectors is linearly independent then their image under the coordinate mapping is also linearly independent.

True, see exercise on page 254 no.25.

- If A is an $n \times n$ matrix and A is invertible then $NulA$ contains infinitely many vectors.

  False, see the IMT on page 267. If A is invertible then $NulA = \{\mathbf{0}\}$.

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7. The following set of vectors, $S = \{t + t^2, 3, 6 + t, t^2\}$, spans $\mathbb{P}_2$. Using the elements of S give a basis for P_2. Use as few computations as necessary, justify your solution.

Using the coordinate mapping we may consider the set in $\mathbb{R}^3$
$S=\left\{\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}3\\0\\0\end{bmatrix},\begin{bmatrix}6\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ As there are more vectors than entries in each vector the set is linearly ~~ind~~ dependent. Using Row Red. I can find a linear dependence relation and remove $\begin{bmatrix}0\\0\\1\end{bmatrix}$.
So $\{t+t^2, 3, 6+t\}$ is a basis for $\mathbb{P}_2$.

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8. The following set of vectors is not a basis for $\mathbb{R}^3$. Show how this set can be expanded to form a basis for $\mathbb{R}^3$.

$$\mathbf{b_1} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\mathbf{b_2} = \begin{bmatrix}2\\0\\2\end{bmatrix}$$

Find a vector $\vec{x}$ which is not a linear combination of $\vec{b}_1$ and $\vec{b}_2$. Then set $\vec{x} = \vec{b}_3$ upon which $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ will be a set of three linearly independent vectors in $\mathbb{R}^3$ and thus a basis, by the Basis Theorem.

It can be shown that $\vec{x} = \begin{bmatrix}6\\6\\6\end{bmatrix}$ is not in Span$\{\vec{b}_1, \vec{b}_2\}$

Thus $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\2\end{bmatrix},\begin{bmatrix}6\\6\\6\end{bmatrix}\right\}$ is a basis, as promised by the Spanning Set Theorem.

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9. Given two bases, $\mathcal{B}, \mathcal{C}$, for the same vector space V, the change of coordinates matrix,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$$

and $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$. (Be careful.)

To do so we must find $\underset{B\leftarrow C}{P}$.

But $\underset{B\leftarrow C}{P} = \left(\underset{C\leftarrow B}{P}\right)^{-1}$. So we find $\left(\underset{B\leftarrow C}{P}\right)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1/2 & -1/2 \\ -3/4 & 5/4 \end{bmatrix}$$

$$[x]_B = \underset{B\leftarrow C}{P} \cdot [x]_C$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ -3/4 & 5/4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{34}{4} \end{bmatrix}.$$

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10. Prove that the null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

See page 227.

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11. Suppose that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is a linearly dependent spanning set for a vector space V. Show that each $\mathbf{w}$ in V can be expressed in more than one way as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$.

Page 254 no 20

As the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans for each $\vec{w} \in V$ we can write $\vec{w}$ as a linear combination of the $\vec{v}_i$'s. That is, $\exists$ scalars k_1, k_2, k_3, k_4 s.t.

$$\vec{w} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3 + k_4 \vec{v}_4$$

Because the set is linearly dependent $\exists$ ~~non~~ scalars, not all zero, s.t.

$$\vec{0} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 .$$

Adding these equations we obtain

$$\vec{w} + \vec{0} = \vec{w} = (k_1 + c_1)\vec{v}_1 + \cdots + (k_4 + c_4)\vec{v}_4$$

And for some i, $1 \le i \le 4$, we must have

$k_i + c_i \neq k_i \implies$ Two ways to write w as a linear combination.