Linear Algebra
Exam 1
Spring 2007

March 15, 2007

Name: SOLUTION KEY (Total 55 points, plus 5 more for Pledged Assignment.)
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted.

WARNING: Please do not say that a matrix A is linearly dependent, rather we say that the columns of A are linearly dependent.

The number in square brackets indicates the value of the problem.

1. **Define:** [4] Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p}$ in $\mathbb{R}^n$ then the **subset of** $\mathbb{R}^n$ **spanned by** $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p}$ is ...

   the collection of all vectors that can be written in the form

   $$c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}$$

   with $c_1, \ldots, c_p$ scalars.

   See page 35 of the text.

2. [6] **Define** what it means for a mapping to be **onto**. Give an example of such a mapping.

   A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

   If $m = n$ then the mapping $\mathbf{x} \mapsto I_n\mathbf{x}$ is onto, where I_n is the identity matrix. (This mapping is also one-to-one.)

   See page 87 of the text.

3. [4] Assuming that T is a linear transformation, find the standard matrix of T, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e_1}$ into $\mathbf{e_1} - \mathbf{2e_2}$, but leaves the vector $\mathbf{e_2}$ unchanged.

Using Theorem 10 on page 83, we recall that the standard matrix is $A = [T(\mathbf{e_1} \ldots T(\mathbf{e_n})]$.

We find here that $T(\mathbf{e_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

And that, $T(\mathbf{e_2}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

And so,

$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

4. [6] Is the following matrix singular? Why, or why not?

$A = \begin{bmatrix} 1 & 6 \\ -1/2 & 2 \end{bmatrix}$

The matrix is not singular, that is, it is non-singular (invertible) since $(1)(2) - (6)(-1/2) \neq 0$.

Compute the inverse of this matrix and use it to solve the equation $A\mathbf{x} = \mathbf{b}$, where

$b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2/5 & -6/5 \\ 1/10 & 1/5 \end{bmatrix}$

and we obtain that $A^{-1}\mathbf{b}$ is equal to

$\mathbf{x} = \begin{bmatrix} -16/5 \\ 7/10 \end{bmatrix}$

5. [8] Is the following set of vectors linearly dependent? If not, give a justification. If so, give a linear dependence relation for them.

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$\mathbf{v_2} = \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix}$$

$$\mathbf{v_3} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

If we consider the matrix, A, whose columns are $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, and perform Gaussian elimination on the augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$. We obtain the following reduced echelon form of

$$A_1 = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

That is, we have x_3 as a free variable - so that the given column vectors are linearly dependent (again we can invoke the IMT). If we let $x_3 = 1$, then $x_1 = x_2 = 3$ and we obtain the linear dependence relation of

$$3\mathbf{v_1} + 3\mathbf{v_2} + \mathbf{v_3} = \mathbf{0}.$$

There are infinitely many other possible linear dependence relations.

6. [$15 - 3each$] **True or False:** Justify each answer by citing an appropriate definition or theorem. If the statement is false and you can provide a counterexample to demonstrate this then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

- If the columns of an $n \times n$ matrix A are linearly independent, then the columns of A span $\mathbb{R}^n$.

  True by the IMT.

- There exists a one-to-one linear transformation mapping $\mathbb{R}^3$ to $\mathbb{R}^2$.

  False. By Theorem 11 (of Chap. 1), T is 1-1 iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. However, the standard matrix of any such transformation is guaranteed a free variable, thus more than the trivial solution.

- If one row in an echelon form of an augmented matrix is [0 0 0 5 0], the the associated linear system is inconsistent.

  False, the variable x_4 could be zero and the system could still be consistent.

- An inconsistent linear system has more than one solution.

  False, by definition it has no solutions.

- The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A.

  False, this is the range.

7. [6] Determine if the following matrices are invertible. Use as few calculations as necessary. Justify your answer.

$$A_1 = \begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 98 & 3 \\ 0 & 0 & 33 & -9 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

The matrix A_1 has 4 pivots. Thus by the invertible matrix theorem it is invertible.

$$A_2 = \begin{bmatrix} 1 & 0 & -2 & 11 \\ 2 & 1 & -4 & 11 \\ 3 & 0 & -6 & 11 \\ 4 & 0 & -8 & 11 \end{bmatrix}$$

For A_2, the third column is a scalar multiple of the first. Therefore the columns of A_2 are not linearly independent, which implies by the invertible matrix theorem that A_2 is NOT invertible.

8. [5] Compute the product $A_2 A_1$ (using the matrices from the previously problem).

$$A_2 A_1 = \begin{bmatrix} 1 & 0 & -59 & 144 \\ 2 & 1 & -20 & 170 \\ 3 & 0 & -177 & 190 \\ 4 & 0 & -236 & 213 \end{bmatrix}$$

Show in as few calculations possible that this product does not equal $A_1 A_2$.

The $(1,1)$ entry in this product is 42, which differs from the $(1,1)$ entry from the above product. Thus, the matrices are not equal.

9. [6] Given $\mathbf{v_1}, \mathbf{v_2} \neq \mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^3$. Further, $\mathbf{v_1}$ is not a scalar multiple of $\mathbf{v_2}$.

- Give a geometric description of the parametric equation $\mathbf{x} = \mathbf{p} + s\mathbf{v_1} + t\mathbf{v_2}$.

  The set of points satisfying this equation is a plane through $\mathbf{p}$, parallel to the plane through the origin containing the vecotrs $\mathbf{v_1}, \mathbf{v_2}$. A drawing would be appropriate to show.

- Given a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Describe the image of the set of vectors satisfying the above parametric equation under T.

  As T is a linear transformation, we can say that

$$T(\mathbf{p} + s\mathbf{v_1} + t\mathbf{v_2}) = T(\mathbf{p}) + sT(\mathbf{v_1}) + tT(\mathbf{v_2}).$$

  If $T(\mathbf{v_1}), T(\mathbf{v_1}) \neq \mathbf{0}$ then the image of this plane is another plane. It is a line if either, but not both, are $\mathbf{0}$. It is the point given by $T(\mathbf{p})$ if both are $\mathbf{0}$.
  This problem generalize the homework problem from Section 1.8, 25.

10. [5] Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Prove the following statement: If T is one-to-one then the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. Do NOT claim this is true by the Invertible Matrix Theorem. (Note that the IMT would only apply if $n = m$.)

    Since T is linear we have that $T(\mathbf{0}) = \mathbf{0}$. If T is one-to-one then by the definition the equation $T(\mathbf{x}) = \mathbf{0}$ has at most one solution and hence only the trivial solution.

    See Theorem 11 on page 88.