(c) Give a careful argument for why $f'(x)$ fails to be continuous on C. Remember that C contains many points besides the endpoints of the intervals that make up $C_1, C_2, C_3, \ldots$.

Let's take inventory of the situation. Our goal is to create a nonintegrable derivative. Our function $f(x)$ is differentiable, and f' fails to be continuous on C. We are not quite done.

Exercise 7.6.18. Why is $f'(x)$ Riemann-integrable on $[0, 1]$?

The reason the Cantor set has measure zero is that, at each stage, 2^{n-1} open intervals of length $1/3^n$ are removed from C_{n-1}. The resulting sum

$$\sum_{n=1}^{\infty} 2^{n-1}\left(\frac{1}{3^n}\right)$$

converges to one, which means that the approximating sets $C_1, C_2, C_3, \ldots$ have total lengths tending to zero. Instead of removing open intervals of length $1/3^n$ at each stage, let's see what happens when we remove intervals of length $1/3^{n+1}$.

Exercise 7.6.19. Show that, under these circumstances, the sum of the lengths of the intervals making up each C_n no longer tends to zero as $n \to \infty$. What is this limit?

If we again take the intersection $\bigcap_{n=0}^{\infty} C_n$, the result is a Cantor-type set with the same topological properties—it is closed, compact and perfect. But a consequence of the previous exercise is that it no longer has measure zero. This is just what we need to define our desired function. By repeating the preceding construction of $f(x)$ on this new Cantor-type set of *positive* measure, we get a differentiable function whose derivative has too many points of discontinuity. By Lebesgue's Theorem, this derivative cannot be integrated using the Riemann integral.

7.7 Epilogue

Riemann's definition of the integral was a modification of Cauchy's integral, which was originally designed for the purpose of integrating continuous functions. In this goal, the Riemann integral was a complete success. For continuous functions at least, the process of integration now stood on its own rigorous footing, defined independently of differentiation. As analysis progressed, however, the dependence of integrability on continuity became problematic. The last example of Section 7.6 highlights one type of weakness: not every derivative can be integrated. Another limitation of the Riemann integral arises in association with limits of sequences of functions. To get a sense of this, let's once again consider Dirichlet's function $g(x)$ introduced in Section 4.1. Recall that $g(x) = 1$ whenever x is rational, and $g(x) = 0$ at every irrational point. Focusing on the interval $[0, 1]$ for a moment, let

$$\{r_1, r_2, r_3, r_4 \ldots\}$$

be an enumeration of the countable number of rational points in this interval. Now, let $g_1(x) = 1$ if $x = r_1$ and define $g_1(x) = 0$ otherwise. Next, define $g_2(x) = 1$ if x is either r_1 or r_2, and let $g_2(x) = 0$ at all other points. In general, for each $n \in \mathbf{N}$, define

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \dots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that each g_n has only a finite number of discontinuities and so is Riemann-integrable with $\int_0^1 g_n = 0$. But we also have $g_n \to g$ pointwise on the interval $[0, 1]$. The problem arises when we remember that Dirichlet's nowhere-continuous function is not Riemann-integrable. Thus, the equation

$$\text{(1)} \qquad \lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$$

fails to hold, not because the values on each side of the equal sign are different but because the value on the right-hand side does not exist. The content of Theorem 7.4.4 is that this equation does hold whenever we have $g_n \to g$ *uniformly*. This is a reasonable way to resolve the situation, but it is a bit unsatisfying because the deficiency in this case is not entirely with the type of convergence but lies in the strength of the Riemann integral. If we could make sense of the right-hand side via some other definition of integration, then maybe equation (1) would actually be true.

Such a definition was introduced by Henri Lebesque in 1901. Generally speaking, Lebesgue's integral is constructed using a generalization of length called the *measure* of a set. In the previous section, we studied sets of *measure zero*. In particular, we showed that the rational numbers in [0,1] (because they are countable) have measure zero. The irrational numbers in [0,1] have measure one. This should not be too surprising because we now have that the measures of these two disjoint sets add up to the length of the interval $[0, 1]$. Rather than chopping up the x-axis to approximate the area under the curve, Lebesgue suggested partitioning the y-axis. In the case of Dirichlet's function g, there are only two range values—zero and one. The integral, according to Lebesgue, could be defined via

$$\begin{aligned} \int_0^1 g &= 1 \cdot [\text{measure of set where } g = 1] + 0 \cdot [\text{measure of set where } g = 0] \\ &= 1 \cdot 0 + 0 \cdot 1 = 0. \end{aligned}$$

With this interpretation of $\int_0^1 g$, equation (1) is now valid!

The Lebesgue integral is presently the standard integral in advanced mathematics. The theory is taught to all graduate students, as well as to many advanced undergraduates, and it is the integral used in most research papers where integration is required. The Lebesgue integral generalizes the Riemann integral in the sense that any function that is Riemann-integrable is Lebesgue-integrable and integrates to the same value. The real strength of the Lebesgue integral is that the class of integrable functions is much larger. Most importantly, this class includes the limits of different types of Cauchy sequences of integrable functions. This leads to a group of extremely important convergence theorems related to equation (1) with hypotheses much weaker than the uniform convergence assumed in Theorem 7.4.4.

Despite its prevalence, the Lebesgue integral does have a few drawbacks. There are functions whose *improper* Riemann integrals exist but that are not Lebesgue-integrable. Another disappointment arises from the relationship between integration and differentiation. Even with the Lebesgue integral, it is still not possible to prove

$$\int_a^b f' = f(b) - f(a)$$

without some additional assumptions on f. Around 1960, a new integral was proposed that can integrate a larger class of functions than either the Riemann integral or the Lebesgue integral and suffers from neither of the preceding weaknesses. Remarkably, this integral is actually a return to Riemann's original technique for defining integration, with some small modifications in how we describe the "fineness" of the partitions. An introduction to the generalized Riemann integral is the topic of Section 8.1.