

## Chapter 7

# The Riemann Integral

### 7.1 Discussion: How Should Integration be Defined?

The Fundamental Theorem of Calculus is a statement about the inverse relationship between differentiation and integration. It comes in two parts, depending on whether we are differentiating an integral or integrating a derivative. Under suitable hypotheses on the functions  $f$  and  $F$ , the Fundamental Theorem of Calculus states that

$$(i) \int_a^b F'(x) dx = F(b) - F(a) \text{ and}$$

$$(ii) \text{ if } G(x) = \int_a^x f(t) dt, \text{ then } G'(x) = f(x).$$

Before we can undertake any type of rigorous investigation of these statements, we need to settle on a definition for  $\int_a^b f$ . Historically, the concept of integration was defined as the inverse process of differentiation. In other words, the integral of a function  $f$  was understood to be a function  $F$  that satisfied  $F' = f$ . Newton, Leibniz, Fermat, and the other founders of calculus then went on to explore the relationship between antiderivatives and the problem of computing areas. This approach is ultimately unsatisfying from the point of view of analysis because it results in a very limited number of functions that can be integrated. Recall that every derivative satisfies the intermediate value property (Darboux's Theorem, Theorem 5.2.7). This means that any function with a jump discontinuity cannot be a derivative. If we want to define integration via antidifferentiation, then we must accept the consequence that a function as simple as

$$h(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } 1 \leq x \leq 2 \end{cases}$$

is not integrable on the interval  $[0, 2]$ .

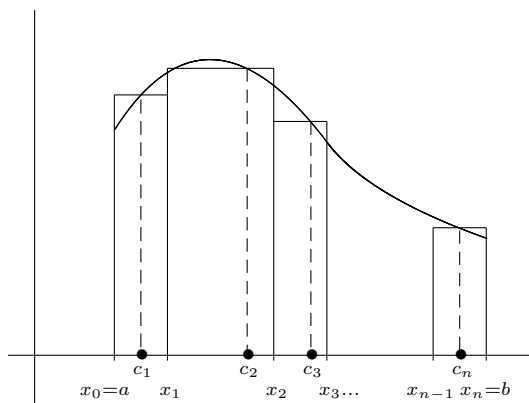


Figure 7.1: A RIEMANN SUM.

A very interesting shift in emphasis occurred around 1850 in the work of Cauchy, and soon after in the work of Bernhard Riemann. The idea was to completely divorce integration from the derivative and instead use the notion of “area under the curve” as a starting point for building a rigorous definition of the integral. The reasons for this were complicated. As we have mentioned earlier (Section 1.2), the concept of *function* was undergoing a transformation. The traditional understanding of a function as a holistic formula such as  $f(x) = x^2$  was being replaced with a more liberal interpretation, which included such bizarre constructions as Dirichlet’s function discussed in Section 4.1. Serving as a catalyst to this evolution was the budding theory of Fourier series (discussed in Section 8.3), which required, among other things, the need to be able to integrate these more unruly objects.

The Riemann integral, as it is called today, is the one usually discussed in introductory calculus. Starting with a function  $f$  on  $[a, b]$ , we partition the domain into small subintervals. On each subinterval  $[x_{k-1}, x_k]$ , we pick some point  $c_k \in [x_{k-1}, x_k]$  and use the  $y$ -value  $f(c_k)$  as an approximation for  $f$  on  $[x_{k-1}, x_k]$ . Graphically speaking, the result is a row of thin rectangles constructed to approximate the area between  $f$  and the  $x$ -axis. The area of each rectangle is  $f(c_k)(x_k - x_{k-1})$ , and so the total area of all of the rectangles is given by the *Riemann sum* (Fig. 7.1)

$$\sum_{k=1}^n f(c_k)(x_k - x_{k-1}).$$

Note that “area” here comes with the understanding that areas below the  $x$ -axis are assigned a negative value.

What should be evident from the graph is that the accuracy of the Riemann-sum approximation seems to improve as the rectangles get thinner. In some

sense, we take the *limit* of these approximating Riemann sums as the width of the individual subintervals of the partitions tends to zero. This limit, if it exists, is Riemann's definition of  $\int_a^b f$ .

This brings us to a handful of questions. Creating a rigorous meaning for the limit just referred to is not too difficult. What will be of most interest to us—and was also to Riemann—is deciding what types of functions can be integrated using this procedure. Specifically, what conditions on  $f$  guarantee that this limit exists?

The theory of the Riemann integral turns on the observation that smaller subintervals produce better approximations to the function  $f$ . On each subinterval  $[x_{k-1}, x_k]$ , the function  $f$  is approximated by its value at some point  $c_k \in [x_{k-1}, x_k]$ . The quality of the approximation is directly related to the difference

$$|f(x) - f(c_k)|$$

as  $x$  ranges over the subinterval. Because the subintervals can be chosen to have arbitrarily small width, this means that we want  $f(x)$  to be close to  $f(c_k)$  whenever  $x$  is close to  $c_k$ . But this sounds like a discussion of continuity! We will soon see that the continuity of  $f$  is intimately related to the existence of the Riemann integral  $\int_a^b f$ .

Is continuity sufficient to prove that the Riemann sums converge to a well-defined limit? Is it necessary, or can the Riemann integral handle a discontinuous function such as  $h(x)$  mentioned earlier? Relying on the intuitive notion of area, it would seem that  $\int_0^2 h = 3$ , but does the Riemann integral reach this conclusion? If so, how discontinuous can a function be before it fails to be integrable? Can the Riemann integral make sense out of something as pathological as Dirichlet's function on the interval  $[0, 1]$ ?

A function such as

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

raises another interesting question. Here is an example of a differentiable function, studied in Section 5.1, where the derivative  $g'(x)$  is *not* continuous. As we explore the class of integrable functions, some attempt must be made to reunite the integral with the derivative. Having defined integration independently of differentiation, we would like to come back and investigate the conditions under which equations (i) and (ii) from the Fundamental Theorem of Calculus stated earlier hold. If we are making a wish list for the types of functions that we want to be integrable, then in light of equation (i) it seems desirable to expect this set to at least contain the set of derivatives. The fact that derivatives are not always continuous is further motivation not to content ourselves with an integral that cannot handle some discontinuities.

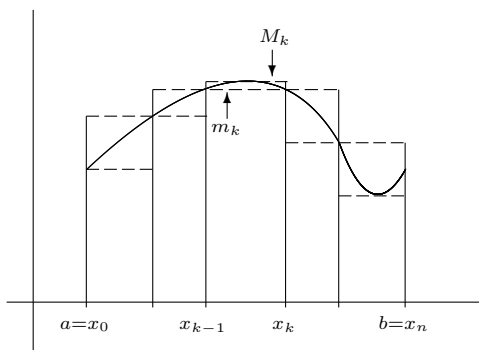


Figure 7.2: UPPER AND LOWER SUMS.

## 7.2 The Definition of the Riemann Integral

Although it has the benefit of some modern polish, the development of the integral presented in this chapter is closely related to the procedure just discussed. In place of Riemann sums, we will construct *upper sums* and *lower sums* (Fig. 7.2), and in place of a limit we will use a supremum and an infimum.

Throughout this section, it is assumed that we are working with a *bounded* function  $f$  on a closed interval  $[a, b]$ , meaning that there exists an  $M > 0$  such that  $|f(x)| \leq M$  for all  $x \in [a, b]$ .

### Partitions, Upper Sums, and Lower Sums

**Definition 7.2.1.** A *partition*  $P$  of  $[a, b]$  is a finite, ordered set

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

For each subinterval  $[x_{k-1}, x_k]$  of  $P$ , let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The *lower sum* of  $f$  with respect to  $P$  is given by

$$L(f, P) = \sum_{k=1}^n m_k(x_k - x_{k-1}).$$

Likewise, we define the *upper sum* of  $f$  with respect to  $P$  by

$$U(f, P) = \sum_{k=1}^n M_k(x_k - x_{k-1}).$$