

6.7 Epilogue

The fact that power series behave so impeccably well under the operations of calculus makes the search for Taylor series expansions a worthwhile enterprise. As it turns out, the traditional list of functions from calculus— $\sin(x)$, $\ln(x)$, $\arccos(x)$, $\sqrt{1+x}$ —all have Taylor series representations that converge on some nontrivial interval to the function from which they were derived. This fact played a major role in the expanding achievements of calculus in the 17th and 18th centuries and understandably led to speculation that every function could be represented in such a fashion. (The term “function” at this time implicitly referred to functions that were infinitely differentiable.) This point of view effectively ended with Cauchy’s discovery in 1821 of the counterexample presented at the end of the previous section. So under what conditions does the Taylor series necessarily converge to the generating function? Lagrange’s Remainder Theorem states that the difference between the Taylor polynomial $S_N(x)$ and the function $f(x)$ is given by

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}.$$

The Ratio Test shows that the $(N+1)!$ term in the denominator grows more rapidly than the x^{N+1} term in the numerator. Thus, if we knew for instance that

$$|f^{(N+1)}(c)| \leq M$$

for all $c \in (-R, R)$ and $N \in \mathbf{N}$, we could be sure that $E_N(x) \rightarrow 0$ and hence that $S_N(x) \rightarrow f(x)$. This is the case for $\sin(x)$, $\cos(x)$, and e^x , whose derivatives do not grow at all as $N \rightarrow \infty$. It is also possible to formulate weaker conditions on the rate of growth of $f^{(N+1)}$ that guarantee convergence.

It is not altogether clear whether Cauchy’s counterexample should come as a surprise. The fact that every previous search for a Taylor series ended in success certainly gives the impression that a power series representation is an intrinsic property of infinitely differentiable functions. But notice what we are saying here. A Taylor series for a function f is constructed from the values of f and its derivatives at the origin. If the Taylor series converges to f on some interval $(-R, R)$, then the behavior of f near zero completely determines its behavior at every point in $(-R, R)$. One implication of this would be that if two functions with Taylor series agree on some small neighborhood $(-\epsilon, \epsilon)$, then these two functions would have to be the same everywhere. When it is put this way, we probably should not expect a Taylor series to always converge back to the function from which it was derived. As we have seen, this is not the case for real-valued functions. What is fascinating, however, is that results of this nature *do* hold for functions of a complex variable. The definition of the derivative looks symbolically the same when the real numbers are replaced by complex numbers, but the implications are profoundly different. In this setting, a function that is differentiable at every point in some open disc must necessarily be infinitely differentiable on this set. This supplies the ingredients to construct

the Taylor series that in every instance converges uniformly on compact sets to the function that generated it.