

defined a continuous nowhere-differentiable function provided $0 < a < 1$ and b was an odd integer satisfying $ab > 1 + 3\pi/2$. The condition on a is easy to understand. If $0 < a < 1$, then $\sum_{n=0}^{\infty} a^n$ is a convergent geometric series, and the forthcoming Weierstrass M-Test (Theorem 6.4.5) can be used to conclude that f is continuous. The restriction on b is more mysterious. In 1916, G.H. Hardy extended Weierstrass' result to include any value of b for which $ab \geq 1$. Without looking at the details of either of these arguments, we nevertheless get a sense that the lack of a derivative is intricately tied to the relationship between the compression factor (the parameter a) and the rate at which the frequency of the oscillations increases (the parameter b).

Exercise 5.4.7. Review the argument for the nondifferentiability of $g(x)$ at nondyadic points. Does the argument still work if we replace $g(x)$ with the summation $\sum_{n=0}^{\infty} (1/2^n)h(3^n x)$? Does the argument work for the function $\sum_{n=0}^{\infty} (1/3^n)h(2^n x)$?

5.5 Epilogue

Far from being an anomaly to be relegated to the margins of our understanding of continuous functions, Weierstrass' example and those like it should actually serve as a guide to our intuition. The image of continuity as a smooth curve in our mind's eye severely misrepresents the situation and is the result of a bias stemming from an overexposure to the much smaller class of differentiable functions. The lesson here is that continuity is a strictly weaker notion than differentiability. In Section 3.6, we alluded to a corollary of the Baire Category Theorem, which asserts that Weierstrass' construction is actually typical of continuous functions. We will see that most continuous functions are nowhere-differentiable, so that it is really the differentiable functions that are the exceptions rather than the rule. The details of how to phrase this observation more rigorously are spelled out in Section 8.2.

To say that the nowhere-differentiable function g constructed in the previous section has “corners” at every point of its domain slightly misses the mark. Weierstrass' original class of nowhere-differentiable functions was constructed from infinite sums of *smooth* trigonometric functions. It is the densely nested oscillating structure that makes the definition of a tangent line impossible. So what happens when we restrict our attention to monotone functions? How nondifferentiable can an increasing function be? Given a finite set of points, it is not difficult to piece together a monotone function which has actual corners—and thus is not differentiable—at each point in the given set. A natural question is whether there exists a continuous, monotone function that is nowhere-differentiable. Weierstrass suspected that such a function existed but only managed to produce an example of a continuous, increasing function which failed to be differentiable on a countable dense set (Exercise 7.5.11). In 1903, the French mathematician Henri Lebesgue (1875–1941) demonstrated that Weierstrass' intuition had failed on this account. Lebesgue proved that a continuous, monotone function would have to be differentiable at “almost” every point in

its domain. To be specific, Lebesgue showed that, for every $\epsilon > 0$, the set of points where such a function fails to be differentiable can be covered by a countable union of intervals whose lengths sum to a quantity less than ϵ . This notion of “zero length,” or “measure zero” as it is called, was encountered in our discussion of the Cantor set and is explored more fully in Section 7.6, where Lebesgue’s substantial contribution to the theory of integration is discussed.

With the relationship between the continuity of f and the existence of f' somewhat in hand, we once more return to the question of characterizing the set of all derivatives. Not every function is a derivative. Darboux’s Theorem forces us to conclude that there are some functions—those with jump discontinuities in particular—that cannot appear as the derivative of some other function. Another way to phrase Darboux’s Theorem is to say that all derivatives must satisfy the intermediate value property. Continuous functions do possess the intermediate value property, and it is natural to ask whether every continuous function is necessarily a derivative. For this smaller class of functions, the answer is yes. The Fundamental Theorem of Calculus, treated in Chapter 7, states that, given a continuous function f , the function $F(x) = \int_a^x f$ satisfies $F' = f$. This does the trick. The collection of derivatives at least contains the continuous functions. The search for a concise characterization of *all* possible derivatives, however, remains largely unsuccessful.

As a final remark, we will see that by cleverly choosing f , this technique of defining F via $F(x) = \int_a^x f$ can be used to produce examples of continuous functions which fail to be differentiable on interesting sets, *provided we can show that $\int_a^x f$ is defined*. The question of just how to define integration became a central theme in analysis in the latter half of the 19th century and has continued on to the present. Much of this story is discussed in detail in Chapter 7 and Section 8.1.