

Definition 4.6.5. Let f be defined on \mathbf{R} , and let $\alpha > 0$. The function f is α -continuous at $x \in \mathbf{R}$ if there exists a $\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta)$ it follows that $|f(y) - f(z)| < \alpha$.

The most important thing to note about this definition is that there is no “for all” in front of the $\alpha > 0$. As we will investigate, adding this quantifier would make this definition equivalent to our definition of continuity. In a sense, α -continuity is a measure of the variation of the function in the neighborhood of a particular point. A function is α -continuous at a point c if there is some interval centered at c in which the variation of the function never exceeds the value $\alpha > 0$.

Given a function f on \mathbf{R} , define D_α to be the set of points where the function f fails to be α -continuous. In other words,

$$D_\alpha = \{x \in \mathbf{R} : f \text{ is not } \alpha\text{-continuous at } x\}.$$

Exercise 4.6.7. Prove that, for a fixed $\alpha > 0$, the set D_α is closed.

The stage is set. It is time to characterize the set of discontinuity for an arbitrary function f on \mathbf{R} .

Theorem 4.6.6. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be an arbitrary function. Then, D_f is an F_σ set.

Proof. Recall that

$$D_f = \{x \in \mathbf{R} : f \text{ is not continuous at } x\}.$$

Exercise 4.6.8. If $\alpha_1 < \alpha_2$, show that $D_{\alpha_2} \subseteq D_{\alpha_1}$.

Exercise 4.6.9. Let $\alpha > 0$ be given. Show that if f is continuous at x , then it is α -continuous at x as well. Explain how it follows that $D_\alpha \subseteq D_f$.

Exercise 4.6.10. Show that if f is not continuous at x , then f is not α -continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}.$$

Because each $D_{\frac{1}{n}}$ is closed, the proof is complete. \square

4.7 Epilogue

Theorem 4.6.6 is only interesting if we can demonstrate that not every subset of \mathbf{R} is in an F_σ set. This takes some effort and was included as an exercise in Section 3.5 on the Baire Category Theorem. Baire’s Theorem states that if \mathbf{R} is written as the countable union of closed sets, then at least one of these sets must contain a nonempty open interval. Now \mathbf{Q} is the countable union of singleton points, and we can view each point as a closed set that obviously contains no

intervals. If the set of irrationals \mathbf{I} were a countable union of closed sets, it would have to be that none of these closed sets contained any open intervals or else they would then contain some rational numbers. But this leads to a contradiction to Baire's Theorem. Thus, \mathbf{I} is not the countable union of closed sets, and consequently it is not an F_σ set. We may therefore conclude that there is no function f that is continuous at every rational point and discontinuous at every irrational point. This should be compared with Thomae's function discussed earlier.

The converse question is interesting as well. Given an arbitrary F_σ set, W.H. Young showed in 1903 that it is always possible to construct a function that has discontinuities precisely on this set. His construction involves the same Dirichlet-type definitions we have seen but is understandably more intricate. By contrast, a function demonstrating the converse for the monotone case is not too difficult to describe. Let

$$D = \{x_1, x_2, x_3, x_4, \dots\}$$

be an arbitrary countable set of real numbers. In order to construct a monotone function that has discontinuities precisely on D , intuitively attach a "weight" of $1/2^n$ to each point $x_n \in D$. Now, define

$$f(x) = \sum_{n: x_n < x} \frac{1}{2^n}$$

where for each $x \in \mathbf{R}$ the sum is intended to be taken over all of those weights corresponding to points to the left of x . (If there are no points in D to the left of x , then set $f(x) = 0$.) Any worries about the order of the sum can be alleviated by observing that the convergence is absolute. It is not too hard to show that the resulting function f is monotone and has jump discontinuities of size $1/2^n$ at each point x_n in D , as desired (Exercise 6.4.8).