Chapter 4

Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

Although it is common practice in calculus courses to discuss continuity before differentiation, historically mathematicians' attention to the concept of continuity came long after the derivative was in wide use. Pierre de Fermat (1601–1665) was using tangent lines to solve optimization problems as early as 1629. On the other hand, it was not until around 1820 that Cauchy, Bolzano, Weierstrass, and others began to characterize continuity in terms more rigorous than prevailing intuitive notions such as "unbroken curves" or "functions which have no jumps or gaps." The basic reason for this two-hundred year waiting period lies in the fact that, for most of this time, the very notion of *function* did not really permit discontinuities. Functions were entities such as polynomials, sines, and cosines, always smooth and continuous over their relevant domains. The gradual liberation of the term function to its modern understanding—a rule associating a unique output to a given input—was simultaneous with 19th century investigations into the behavior of infinite series. Extensions of the power of calculus were intimately tied to the ability to represent a function $f(x)$ as a limit of polynomials (called a *power series*) or as a limit of sums of sines and cosines (called a *trigonometric* or *Fourier* series). A typical question for Cauchy and his contemporaries was whether the continuity of the limiting polynomials or trigonometric functions necessarily implied that the limit f would also be continuous.

Sequences and series of functions are the topics of Chapter 6. What is relevant at this moment is that we realize why the issue of finding a rigorous definition for continuity finally made its way to the fore. Any significant progress on the question of whether the limit of continuous functions is continuous (for

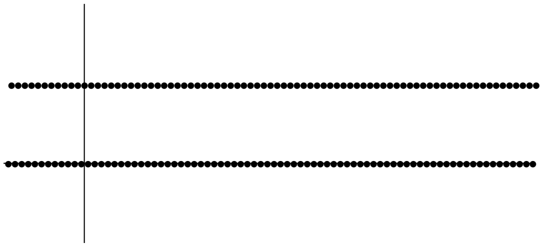

Figure 4.1: Dirichlet's Function, $g(x)$.

Cauchy and for us) necessarily depends on a definition of continuity that does not rely on imprecise notions such as "no holes" or "gaps." With a mathematically unambiguous definition for the limit of a sequence in hand, we are well on our way toward a rigorous understanding of continuity.

Given a function f with domain $A \subseteq \mathbf{R}$, we want to define continuity at a point $c \in A$ to mean that if $x \in A$ is chosen *near* c, then $f(x)$ will be *near* $f(c)$. Symbolically, we will say f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

The problem is that, at present, we only have a definition for the limit of a sequence, and it is not entirely clear what is meant by $\lim_{x \to c} f(x)$. The subtleties that arise as we try to fashion such a definition are well-illustrated via a family of examples, all based on an idea of the prominent German mathematician, Peter Lejeune Dirichlet. Dirichlet's idea was to define a function g in a piecewise manner based on whether or not the input variable x is rational or irrational. Specifically, let

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

The intricate way that $\mathbf{Q}$ and $\mathbf{I}$ fit inside of $\mathbf{R}$ makes an accurate graph of g technically impossible to draw, but Figure 4.1 illustrates the basic idea.

Does it make sense to attach a value to the expression $\lim_{x \to 1/2} g(x)$? One idea is to consider a sequence $(x_n) \to 1/2$. Using our notion of the limit of a sequence, we might try to define $\lim_{x \to 1/2} g(x)$ as simply the limit of the sequence $g(x_n)$. But notice that this limit depends on how the sequence (x_n) is chosen. If each x_n is rational, then

$$\lim_{n \to \infty} g(x_n) = 1.$$

On the other hand, if x_n is irrational for each n, then

$$\lim_{n \to \infty} g(x_n) = 0.$$

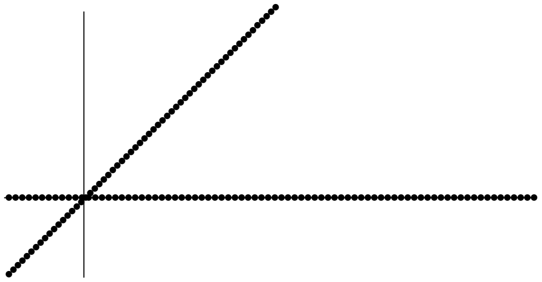

Figure 4.2: Modified Dirichlet Function, $h(x)$.

This unacceptable situation demands that we work harder on our definition of functional limits. Generally speaking, we want the value of $\lim_{x\to c} g(x)$ to be independent of how we approach c. In this particular case, the definition of a functional limit that we agree on should lead to the conclusion that

$$\lim_{x\to 1/2} g(x) \qquad \text{does not exist.}$$

Postponing the search for formal definitions for the moment, we should nonetheless realize that Dirichlet's function is not continuous at $c = 1/2$. In fact, the real significance of this function is that there is nothing unique about the point $c = 1/2$. Because both $\mathbf{Q}$ and $\mathbf{I}$ (the set of irrationals) are dense in the real line, it follows that for any $z \in \mathbf{R}$ we can find sequences $(x_n) \subseteq \mathbf{Q}$ and $(y_n) \subseteq \mathbf{I}$ such that

$$\lim x_n = \lim y_n = z.$$

(See Example 3.2.9 (iii).) Because

$$\lim g(x_n) \neq \lim g(y_n),$$

the same line of reasoning reveals that $g(x)$ is not continuous at z. In the jargon of analysis, Dirichlet's function is a *nowhere-continuous* function on $\mathbf{R}$.

What happens if we adjust the definition of $g(x)$ in the following way? Define a new function h (Fig. 4.2) on $\mathbf{R}$ by setting

$$h(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

If we take c different from zero, then just as before we can construct sequences $(x_n) \to c$ of rationals and $(y_n) \to c$ of irrationals so that

$$\lim h(x_n) = c \qquad \text{and} \qquad \lim h(y_n) = 0.$$

Thus, h is not continuous at every point $c \neq 0$.

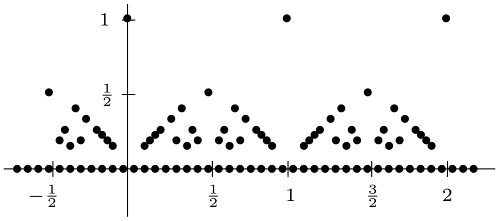

Figure 4.3: THOMAE'S FUNCTION, $t(x)$.

If $c = 0$, however, then these two limits are both equal to $h(0) = 0$. In fact, it appears as though no matter how we construct a sequence (z_n) converging to zero, it will always be the case that $\lim h(z_n) = 0$. This observation goes to the heart of what we want functional limits to entail. To assert that

$$\lim_{x \to c} h(x) = L$$

should imply that

$$h(z_n) \to L \quad \textit{for all} \text{ sequences} \quad (z_n) \to c.$$

For reasons not yet apparent, it is beneficial to fashion the definition for functional limits in terms of neighborhoods constructed around c and L. We will quickly see, however, that this topological formulation is equivalent to the sequential characterization we have arrived at here.

To this point, we have been discussing continuity of a function at a particular point in its domain. This is a significant departure from thinking of continuous functions as curves that can be drawn without lifting the pen from the paper, and it leads to some fascinating questions. In 1875, K.J. Thomae discovered the function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

If $c \in \mathbf{Q}$, then $t(c) > 0$. Because the set of irrationals is dense in $\mathbf{R}$, we can find a sequence (y_n) in $\mathbf{I}$ converging to c. The result is that

$$\lim t(y_n) = 0 \neq t(c),$$

and Thomae's function (Fig. 4.3) fails to be continuous at any rational point.

The twist comes when we try this argument on some irrational point in the domain such as $c = \sqrt{2}$. All irrational values get mapped to zero by t, so the natural thing would be to consider a sequence (x_n) of rational numbers that converges to $\sqrt{2}$. Now, $\sqrt{2} \approx 1.414213\ldots$ so a good start on a particular sequence of rational approximations for $\sqrt{2}$ might be

$$\left(1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots\right).$$

But notice that the denominators of these fractions are getting larger. In this case, the sequence $t(x_n)$ begins,

$$\left(1, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \frac{1}{5000}, \frac{1}{100000}, \ldots\right)$$

and is fast approaching $0 = t(\sqrt{2})$. We will see that this always happens. The closer a rational number is chosen to a fixed irrational number, the larger its denominator must necessarily be. As a consequence, Thomae's function has the bizarre property of being continuous at every irrational point on $\mathbf{R}$ and discontinuous at every rational point.

Is there an example of a function with the opposite property? In other words, does there exist a function defined on all of $\mathbf{R}$ that is continuous on $\mathbf{Q}$ but fails to be continuous on $\mathbf{I}$? Can the set of discontinuities of a particular function be arbitrary? If we are given some set $A \subseteq \mathbf{R}$, is it always possible to find a function that is continuous only on the set A^c? In each of the examples in this section, the functions were defined to have erratic oscillations around points in the domain. What conclusions can we draw if we restrict our attention to functions that are somewhat less volatile? One such class is the set of so-called *monotone* functions, which are either increasing or decreasing on a given domain. What might we be able to say about the set of discontinuities of a monotone function on $\mathbf{R}$?

4.2 Functional Limits

Consider a function $f : A \to \mathbf{R}$. Recall that a limit point c of A is a point with the property that every ϵ-neighborhood $V_\epsilon(c)$ intersects A in some point other than c. Equivalently, c is a limit point of A if and only if $c = \lim x_n$ for some sequence $(x_n) \subseteq A$ with $x_n \neq c$. It is important to remember that limit points of A do not necessarily belong to the set A unless A is closed.

If c is a limit point of the domain of f, then, intuitively, the statement

$$\lim_{x \to c} f(x) = L$$

is intended to convey that values of $f(x)$ get arbitrarily close to L as x is chosen closer and closer to c. The issue of what happens when $x = c$ is irrelevant from the point of view of functional limits. In fact, c need not even be in the domain of f.

The structure of the definition of functional limits follows the "challenge–response" pattern established in the definition for the limit of a sequence. Recall