Exercise 3.5.9. Decide whether the following sets are dense in **R**, nowhere-dense in **R**, or somewhere in between.

(a) $A = \mathbf{Q} \cap [0, 5]$.
(b) $B = \{1/n : n \in \mathbf{N}\}$.
(c) the set of irrationals.
(d) the Cantor set.

We can now restate Theorem 3.5.2 in a slightly more general form.

Theorem 3.5.4 (Baire's Theorem). *The set of real numbers* **R** *cannot be written as the countable union of nowhere-dense sets.*

Proof. For contradiction, assume that $E_1, E_2, E_3, \ldots$ are each nowhere-dense and satisfy $\mathbf{R} = \bigcup_{n=1}^{\infty} E_n$.

Exercise 3.5.10. Finish the proof by finding a contradiction to the results in this section.

□

3.6 Epilogue

Baire's Theorem is yet another statement about the size of **R**. We have already encountered several ways to describe the sizes of infinite sets. In terms of cardinality, countable sets are relatively small whereas uncountable sets are large. We also briefly discussed the concept of "length," or "measure," in Section 3.1. Baire's Theorem offers a third perspective. From this point of view, nowhere-dense sets are considered to be "thin" sets. Any set that is the countable union—i.e., a not very large union—of these small sets is called a "meager" set or a set of "first category." A set that is not of first category is of "second category." Intuitively, sets of the second category are the "fat" subsets. The Baire Category Theorem, as it is often called, states that **R** is of second category.

There is a significance to the Baire Category Theorem that is difficult to appreciate at the moment because we are only seeing a special case of this result. The real numbers are an example of a *complete metric space*. Metric spaces are discussed in some detail in Section 8.2, but here is the basic idea. Given a set of mathematical objects such as real numbers, points in the plane or continuous functions defined on [0,1], a "metric" is a rule that assigns a "distance" between two elements in the set. In **R**, we have been using $|x-y|$ as the distance between the real numbers x and y. The point is that if we can create a satisfactory notion of "distance" on these other spaces (we will need the triangle inequality to hold, for instance), then the concepts of convergence, Cauchy sequences, and open sets, for example, can be naturally transferred over. A complete metric space is any set with a suitably defined metric in which Cauchy sequences have limits. We have spent a good deal of time discussing the fact that **R** is a complete metric space whereas **Q** is not.

The Baire Category Theorem in its more general form states that *any* complete metric space must be too large to be the countable union of nowhere-dense subsets. One particularly interesting example of a complete metric space is the set of continuous functions defined on the interval $[0, 1]$. (The distance between two functions f and g in this space is defined to be $\sup |f(x) - g(x)|$, where $x \in [0, 1]$.) Now, in this space we will see that the collection of continuous functions that are differentiable at even one point *can* be written as the countable union of nowhere-dense sets. Thus, a fascinating consequence of Baire's Theorem in this setting is that *most continuous functions do not have derivatives at any point.* Chapter 5 concludes with a construction of one such function. This odd situation mirrors the roles of $\mathbf{Q}$ and $\mathbf{I}$ as subsets of $\mathbf{R}$. Just as the familiar rational numbers constitute a minute proportion of the real line, the differentiable functions of calculus are exceedingly atypical of continuous functions in general.