This particular form of the product, examined earlier in Exercise 2.8.7, is called the *Cauchy product* of two series. Although there is something algebraically natural about writing the product in this form, it may very well be that computing the value of the sum is more easily done via one or the other iterated summation. The question remains, then, as to how the value of the Cauchy product—if it exists—is related to these other values of the double sum. If the two series being multiplied converge absolutely, it is not too difficult to prove that the sum may be computed in whatever way is most convenient.

Exercise 2.8.8. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the set

$$\left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_i b_j| : m, n \in \mathbf{N} \right\}$$

is bounded. Use this to show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and use the Algebraic Limit Theorem to show that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

2.9 Epilogue

Theorems 2.7.10 and 2.8.1 make it clear that absolute convergence is an extremely desirable quality to have when manipulating series. On the other hand, the situation for conditionally convergent series is delightfully pathological. In the case of rearrangements, not only are they no longer guaranteed to converge to the same limit, but in fact if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for *any* $r \in \mathbf{R}$ there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to r. To see why, let's look again at the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

The negative terms taken alone form the series $\sum_{n=1}^{\infty}(-1)/2n$. The partial sums of this series are precisely $-1/2$ the partial sums of the harmonic series, and so march off (at half speed) to negative infinity. A similar argument shows that the sum of positive terms $\sum_{n=1}^{\infty} 1/(2n-1)$ also diverges to infinity. It is not too difficult to argue that this situation is *always* the case for conditionally convergent series (Exercise 2.7.3). Now, let r be some proposed limit, which, for the sake of this argument, we take to be positive. The idea is to take as many positive terms as necessary to form the first partial sum greater than r. We then add negative terms until the partial sum falls below r, at which point we switch back to positive terms. The fact that there is no bound on the sums of either the positive terms or the negative terms allows this process to continue indefinitely. The fact that the terms themselves tend to zero is enough to guarantee that the partial sums, when constructed in this manner, indeed converge to r as they oscillate around this target value.

Perhaps the best way to summarize the situation is to say that the hypothesis of absolute convergence essentially allows us to treat infinite sums as though they were finite sums. This assessment extends to double sums as well, although there are a few subtleties to address. In the case of products, we showed in Exercise 2.8.8 that the Cauchy product of two absolutely convergent infinite series converges to the product of the two factors, but in fact the same conclusion follows if we only have absolute convergence in one of the two original series. In the notation of Exercise 2.8.8, if $\sum a_n$ converges absolutely to A, and if $\sum b_n$ converges (perhaps conditionally) to B, then the Cauchy product $\sum d_k = AB$. On the other hand, if both $\sum a_n$ and $\sum b_n$ converge conditionally, then it is possible for the Cauchy product to diverge. Squaring $\sum(-1)^n/\sqrt{n}$ provides an example of this phenomenon. Of course, it is also possible to find $\sum a_n = A$ conditionally and $\sum b_n = B$ conditionally whose Cauchy product $\sum d_k$ converges. If this is the case, then the convergence is to the right value, namely $\sum d_k = AB$. A proof of this last fact will be offered in Chapter 6, where we undertake the study of *power series*. Here is the connection. A power series has the form $a_0 + a_1x + a_2x^2 + \cdots$. If we multiply two power series together as though they were polynomials, then when we collect common powers of x the result is

$$\begin{aligned}
&(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots) \\
&= \quad a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots \\
&= \quad d_0 + d_1x + d_2x^2 + \cdots ,
\end{aligned}$$

which is the Cauchy product of $\sum a_nx^n$ and $\sum b_nx^n$. (The index starts with $n = 0$ rather than $n = 1$.) Upcoming results about the good behavior of power series will lead to a proof that convergent Cauchy products sum to the proper value. In the other direction, Exercise 2.8.8 will be useful in establishing a theorem about the product of two power series.