Chapter 1

The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

Toward the end of his distinguished career, the renowned British mathematician G.H. Hardy eloquently laid out a justification for a life of studying mathematics in *A Mathematician's Apology*, an essay first published in 1940. At the center of Hardy's defense is the thesis that mathematics is an aesthetic discipline. For Hardy, the applied mathematics of engineers and economists held little charm. "Real mathematics," as he referred to it, "must be justified as art if it can be justified at all."

To help make his point, Hardy includes two theorems from classical Greek mathematics, which, in his opinion, possess an elusive kind of beauty that, although difficult to define, is easy to recognize. The first of these results is Euclid's proof that there are an infinite number of prime numbers. The second result is the discovery, attributed to the school of Pythagoras from around 500 B.C., that $\sqrt{2}$ is irrational. It is this second theorem that demands our attention. (A course in number theory would focus on the first.) The argument uses only arithmetic, but its depth and importance cannot be overstated. As Hardy says, "[It] is a 'simple' theorem, simple both in idea and execution, but there is no doubt at all about [it being] of the highest class. [It] is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on [it]."

Theorem 1.1.1. *There is no rational number whose square is 2.*

Proof. A rational number is any number that can be expressed in the form p/q, where p and q are integers. Thus, what the theorem asserts is that no matter how p and q are chosen, it is never the case that $(p/q)^2 = 2$. The line of attack is indirect, using a type of argument referred to as a proof by contradiction. The idea is to assume that there *is* a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false.

And so assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \tag{1}$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, equation (1) implies

$$p^2 = 2q^2. \tag{2}$$

From this, we can see that the integer p^2 is an even number (it is divisible by 2), and hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute $2r$ for p in equation (2), then a little algebra yields the relationship

$$2r^2 = q^2.$$

But now the absurdity is at hand. This last equation implies that q^2 is even, and hence q must also be even. Thus, we have shown that p and q are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that equation (1) *cannot* hold for any integers p and q, and thus the theorem is proved. □

A component of Hardy's definition of beauty in a mathematical theorem is that the result have lasting and serious implications for a network of other mathematical ideas. In this case, the ideas under assault were the Greeks' understanding of the relationship between geometric *length* and arithmetic *number*. Prior to the preceding discovery, it was an assumed and commonly used fact that, given two line segments $\overline{AB}$ and $\overline{CD}$, it would always be possible to find a third line segment whose length divides evenly into the first two. In modern terminology, this is equivalent to asserting that the length of $\overline{CD}$ is a rational multiple of the length of $\overline{AB}$. Looking at the diagonal of a unit square (Fig. 1.1), it now followed (using the Pythagorean Theorem) that this was not always the case. Because the Pythagoreans implicitly interpreted number to mean rational number, they were forced to accept that number was a strictly weaker notion than length.

Rather than abandoning arithmetic in favor of geometry (as the Greeks seem to have done), our resolution to this limitation is to strengthen the concept of number by moving from the rational numbers to a larger number system. From a modern point of view, this should seem like a familiar and somewhat natural phenomenon. We begin with the *natural numbers*

$$\mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

The influential German mathematician Leopold Kronecker (1823–1891) once asserted that "The natural numbers are the work of God. All of the rest is

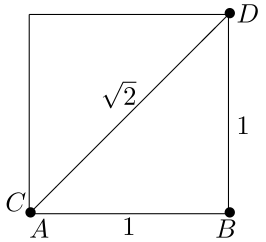

Figure 1.1: $\sqrt{2}$ EXISTS AS A GEOMETRIC LENGTH.

the work of mankind." Debating the validity of this claim is an interesting conversation for another time. For the moment, it at least provides us with a place to start. If we restrict our attention to the natural numbers $\mathbf{N}$, then we can perform addition perfectly well, but we must extend our system to the *integers*

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

if we want to have an additive identity (zero) and the additive inverses necessary to define subtraction. The next issue is multiplication and division. The number 1 acts as the multiplicative identity, but in order to define division we need to have multiplicative inverses. Thus, we extend our system again to the *rational numbers*

$$\mathbf{Q} = \left\{ \text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0 \right\}.$$

Taken together, the properties of $\mathbf{Q}$ discussed in the previous paragraph essentially make up the definition of what is called a *field.* More formally stated, a field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the familiar distributive property $a(b+c) = ab+ac$. There must be an additive identity, and every element must have an additive inverse. Finally, there must be a multiplicative identity, and multiplicative inverses must exist for all nonzero elements of the field. Neither $\mathbf{Z}$ nor $\mathbf{N}$ is a field. The finite set $\{0, 1, 2, 3, 4\}$ is a field when addition and multiplication are computed modulo 5. This is not immediately obvious but makes an interesting exercise (Exercise 1.3.1).

The set $\mathbf{Q}$ also has a natural *order* defined on it. Given any two rational numbers r and s, exactly one of the following is true:

$$r < s, \quad r = s, \quad \text{or} \quad r > s.$$

This ordering is transitive in the sense that if $r < s$ and $s < t$, then $r < t$, so we are conveniently led to a mental picture of the rational numbers as being laid out from left to right along a number line. Unlike $\mathbf{Z}$, there are no intervals of empty space. Given any two rational numbers $r < s$, the rational number

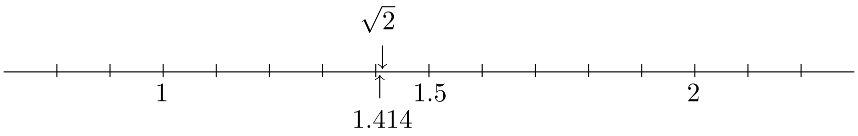

Figure 1.2: Approximating $\sqrt{2}$ with rational numbers.

$(r+s)/2$ sits halfway in between, implying that the rational numbers are densely nestled together.

With the field properties of **Q** allowing us to safely carry out the algebraic operations of addition, subtraction, multiplication, and division, let's remind ourselves just what it is that **Q** is lacking. By Theorem 1.1.1, it is apparent that we cannot always take square roots. The problem, however, is actually more fundamental than this. Using only rational numbers, it is possible to *approximate* $\sqrt{2}$ quite well (Fig. 1.2). For instance, $1.414^2 = 1.999396$. By adding more decimal places to our approximation, we can get even closer to a value for $\sqrt{2}$, but, even so, we are now well aware that there is a "hole" in the rational number line where $\sqrt{2}$ ought to be. Of course, there are quite a few other holes—at $\sqrt{3}$ and $\sqrt{5}$, for example. Returning to the dilemma of the ancient Greek mathematicians, if we want every length along the number line to correspond to an actual number, then another extension to our number system is in order. Thus, to the chain $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q}$ we append the *real numbers* **R**.

The question of how to actually construct **R** from **Q** is rather complicated business. It is discussed in Section 1.3, and then again in more detail in Section 8.4. For the moment, it is not too inaccurate to say that **R** is obtained by filling in the gaps in **Q**. Wherever there is a hole, a new *irrational* number is defined and placed into the ordering that already exists on **Q**. The real numbers are then the union of these irrational numbers together with the more familiar rational ones. What properties does the set of irrational numbers have? How do the sets of rational and irrational numbers fit together? Is there a kind of symmetry between the rationals and the irrationals, or is there some sense in which we can argue that one type of real number is more common than the other? The one method we have seen so far for generating examples of irrational numbers is through square roots. Not too surprisingly, other roots such as $\sqrt[3]{2}$ or $\sqrt[5]{3}$ are most often irrational. Can all irrational numbers be expressed as algebraic combinations of nth roots and rational numbers, or are there still other irrational numbers beyond those of this form?

1.2 Some Preliminaries

The vocabulary necessary for the ensuing development comes from set theory and the theory of functions. This should be familiar territory, but a brief review