PH 500 Problem Set #4

We've established the formalism for time evolution. In this problem set, we apply it more concretely to the two-state system. Again, the things I want you to do are in **bold**.

1. For working with 2 by 2 matrices, it will be very useful to establish some properties of the matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ we defined in an earlier problem set. These are called the *Pauli matrices*. They are (one choice of) the three linearly independent traceless 2 by 2 Hermitian matrices. It is not a coincidence that the number of Pauli matrices is the same as the dimension of space, so we've labelled them with x, y and z.

   Show that

$$
\begin{aligned}
\hat{\sigma}_x\hat{\sigma}_y &= i\hat{\sigma}_z & \qquad \hat{\sigma}_y\hat{\sigma}_x &= -i\hat{\sigma}_z \\
\hat{\sigma}_y\hat{\sigma}_z &= i\hat{\sigma}_x & \qquad \hat{\sigma}_z\hat{\sigma}_y &= -i\hat{\sigma}_x \\
\hat{\sigma}_z\hat{\sigma}_x &= i\hat{\sigma}_y & \qquad \hat{\sigma}_x\hat{\sigma}_z &= -i\hat{\sigma}_y
\end{aligned} \tag{1}
$$

   Note also that $\hat{\sigma}_i^2 = \hat{1}$ for all the Pauli matrices. These results can be summarized by the equation

$$
\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij} + i\sum_{k=1}^{3}\epsilon_{ijk}\hat{\sigma}_k \tag{2}
$$

   where ϵ_{ijk} is the *Levi-Civita symbol*. It is defined such that it changes sign when any two indices are exchanged and $\epsilon_{123} = 1$. Thus $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$ and $\epsilon_{231} = \epsilon_{312} = 1$, and if any two of the indices are equal it is zero. This is a useful symbol — for example you can check that the usual cross product of two real 3-dimensional vectors is

$$
\boldsymbol{a} \times \boldsymbol{b} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}a_i b_j \hat{k} \tag{3}
$$

   where $\hat{\boldsymbol{k}}$ is $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ or $\hat{\boldsymbol{z}}$ for $k = 1$, $k = 2$, or $k = 3$ respectively.

   For any real 3-vector $\boldsymbol{v}$, define

$$
\boldsymbol{v} \cdot \hat{\boldsymbol{\sigma}} = v_x\hat{\sigma}_x + v_y\hat{\sigma}_y + v_z\hat{\sigma}_z \tag{4}
$$

   which is a Hermitian matrix. **Show that for any two real 3-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,**

$$
(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{a})(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}\hat{1} + i(\boldsymbol{a} \times \boldsymbol{b}) \cdot \hat{\boldsymbol{\sigma}} \tag{5}
$$

   where $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ are the usual dot and cross products for 3-dimensional vectors.

2. **Show that if the real 3-vector $\boldsymbol{u}$ has $|\boldsymbol{u}| = 1$, then**

$$e^{i\theta \boldsymbol{u}\cdot\hat{\boldsymbol{\sigma}}} = \hat{1}\cos\theta + i(\boldsymbol{u}\cdot\hat{\boldsymbol{\sigma}})\sin\theta \tag{6}$$

3. The general case. **Start in the arbitrary initial state**

$$|\psi(t=0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

where a and b are complex numbers and take the Hamiltonian as

$$\hat{\mathcal{H}} = -k\boldsymbol{v}\cdot\hat{\boldsymbol{\sigma}} \tag{8}$$

where $\boldsymbol{v}$ is a real, constant, unit 3-vector and k is a constant. Find $|\psi(t)\rangle$. This is the most general 2 by 2 Hamiltonian, because the only other term we could add that would keep $\hat{\mathcal{H}}$ Hermitian would be proportional to the identity matrix, which we have already shown does not affect the dynamics.

4. Having solved this system, we can now turn to interpreting the physics. There are many problems where we can isolate aspects of the system corresponding to a two–state system, including the neutral kaon system, neutrino oscillations, and the ammonia molecule. But the most common is the spin of an electron. Even at rest, electrons possess an intrinsic angular momentum. This is a purely quantum effect (whose origins we will not go into here). As a result, the electron has an intrinsic magnetic dipole moment, as if it were a tiny loop of current. The energy of a magnetic dipole in a magnetic field is $U = -\boldsymbol{M}\cdot\boldsymbol{B}$ where $\boldsymbol{M}$ is the dipole moment, indicating that the loop of current wants to lie perpendicular to the magnetic field. If we think of this current as coming from a particle running in a loop, the magnetic dipole moment is proportional to the angular momentum of the particle, multiplied by some charge-to-mass ratio γ, called the *gyromagnetic ratio.* [1]

   The *intrinsic angular momentum* of the electron, which is usually called the *spin*, creates a magnetic moment in much the same way. The spin is given by the vector of operators $\hat{\boldsymbol{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}$. Then the Hamiltonian is just[2]

$$\hat{\mathcal{H}} = -\gamma\hat{\boldsymbol{S}}\cdot\boldsymbol{B} = -\gamma\frac{\hbar}{2}\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{B} \tag{9}$$

[1] Classically, this ratio is just $\frac{q}{2m}$ if the charge and mass are distributed the same way. For example, if we take a uniformly charged sphere with total charge q, and uniform mass density with total mass m, its magnetic dipole moment is $\frac{q}{2m}$ times its angular momentum.

[2] For the electron spin, the gyromagnetic ratio γ is roughly *twice* the usual value, $g\frac{e}{2m_e}$ with $g \approx 2$. Predicting this result precisely is a triumph of *relativistic* quantum mechanics. In fact, this quantity probably represents the most precise agreement between theory and experiment in the history of science: Theory and experiment agree to one part per trillion! (This is roughly comparable to measuring the distance from Vermont to California accurately to the thickness of a human hair.)

So to understand the physics of this system better, we need to start by understanding angular momentum. Note that the different components of spin *do not commute with each other*. So we cannot measure them all simultaneously. However, using the result you have shown earlier, we know that they all have the same eigenvalues, $\pm\hbar/2$. So a measurement of the spin along any direction will always yield one of these values — "spin up" or "spin down." Note that the *state* corresponding to spin down is *not the negative* of the *state* corresponding to spin up — rather, the two are *orthogonal*. So, roughly, two results that differ by a 180° rotation in real 3-space correspond to vectors at a 90° angle in the 2–dimensional complex vectorspace of states! We'll see this strange result in more detail later.

Another useful operator is $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$.

Show that $\hat{S}^2$ commutes with $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$, and find the possible results of a measurement of $\hat{S}^2$. (In this case, this calculation is somewhat trivial, but this result extends to a number of other situations we'll run into later on.)

5. Angular momentum is intimately connected with *rotations*. Let's represent a rotation by a real 3-vector $\vec{\boldsymbol{\theta}}$, whose direction is the axis of the rotation and whose magnitude is the angle of the rotation. Then the implementation of this rotation on a state is by the unitary transformation

$$|\psi\rangle \to e^{-i\vec{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{S}}/\hbar}|\psi\rangle = e^{-i\vec{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\sigma}}/2}|\psi\rangle \tag{10}$$

The factor of $\frac{1}{2}$ in the exponent reflects the difference between angles in real space and state space that we noted earlier.

Start with the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with definite spin up in the z direction. Find the state that results from rotation by an angle θ around the x axis. Compute the expectation values of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ in this state. Compare the result to your classical intuition for 90° and 180° rotations. What happens to the state when $\theta = 2\pi$?

6. Acting on the Hamiltonian (or any operator), a rotation is given by

$$\hat{\mathcal{H}} \to e^{\frac{-i\vec{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{S}}}{\hbar}}\hat{\mathcal{H}}e^{\frac{i\vec{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{S}}}{\hbar}} \tag{11}$$

You can think of this simply as the operator that rotates the state back the way it was, applies the old operator, and rotates the state forward again. If we get the same Hamiltonian back, we say that the Hamiltonian is *rotation invariant*. (In this system, rotationally invariant Hamiltonians are not very interesting, since they would have to have zero magnetic field: once we pick a direction for the magnetic field we have broken rotation invariance. But with larger systems we will be able to construct nontrivial rotationally invariant Hamiltonians. In our case, the Hamiltonian is invariant under rotations around the axis parallel to the magnetic field direction.) **Show that, in general, if a Hamiltonian is invariant under rotations around a particular axis $\hat{\boldsymbol{n}}$, then the corresponding angular momentum $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{S}}$ around that axis is conserved.** Hint: Consider an infinitesimal rotation, and use the Taylor expansion of the rotation operator. Remember that if two quantities are equal as functions of θ, they must have the same Taylor series in θ.

This is an example of *Noether's theorem*, which holds in both classical and quantum physics: whenever you have a *symmetry* (such as rotation invariance), there is a corresponding *conservation law* (such as angular momentum). Some examples are:

Symmetry	Conserved Quantity
time translation	energy
space translation	momentum
rotation	angular momentum
phase rotations of quantum states	electric charge

7. We can form *raising and lowering operators* from the other two components of $\hat{\boldsymbol{S}}$:

$$\begin{aligned} \hat{S}^{+} &= \hat{S}_x + i\hat{S}_y \\ \hat{S}^{-} &= \hat{S}_x - i\hat{S}_y \end{aligned} \tag{12}$$

Find what happens when you act with each of these operators on each of the eigenvectors of $\hat{S}_z$. Explain why these operators are aptly named.