An Introduction to Solitons and Oscillons

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January 28, 2010

Linearity and Nonlinearity

Most waves we encounter in physics are linear (or at least we treat them that way): They obey superposition. So we can analyze the behavior of each wave mode individually (Fourier).

For example:

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 4\pi\rho \qquad\qquad & \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad\qquad & \nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{J} \end{aligned}$$

Such waves generally disperse. Each wave follows its own path.

Nonlinear theories can be more interesting, but harder to describe:

- Waves interact, so we must consider all modes at once.
- We can get new structures due to these interactions: clumps of waves held together by their own interactions.

A scalar field

Consider a simpler system: scalar instead of vector field. Also go to one space dimension. Example: waves on a string.

The equation of motion is (Here $\dot{\phi} = \frac{\partial \phi}{\partial t}$, $\phi' = \frac{\partial \phi}{\partial x}$):

$$\underbrace{\ddot{\phi}(x,t)}_{\text{acceleration}} = \underbrace{\phi''(x,t)}_{\substack{\text{restoring force}\\ \text{due to stretching}}} - \underbrace{U'(\phi(x,t))}_{\substack{\text{force from}\\ \text{additional potential}}}$$

The energy is

$$U = \int dx \Big(\underbrace{\frac{1}{2}(\dot{\phi})^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}(\phi')^2}_{\substack{\text{potential energy}\\ \text{due to stretching}}} + \underbrace{U(\phi(x,t))}_{\text{additional potential energy}} \Big)$$

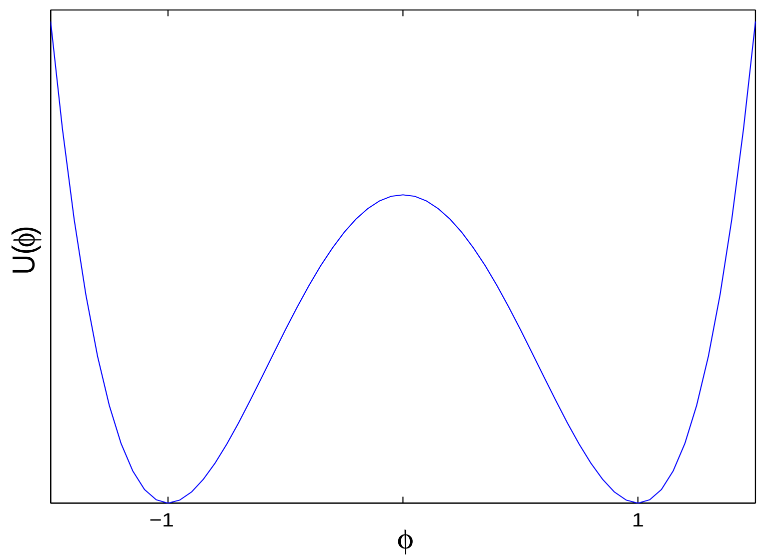
A quadratic term in the potential, $U_{\text{quadratic}} = \frac{1}{2}m^2\phi^2$, still yields linear equations of motion, with the dispersion relation $\omega = \sqrt{k^2 + m^2}$. This corresponds to a mass for our field.

The kink

Consider the potential $U(\phi) = \frac{m^2}{8}\left(\phi^2 - 1\right)^2$:

This is a "double-well" potential with minima at $\phi = \pm 1$. The (nonlinear) field equation is

$$\ddot{\phi}(x,t) = \phi''(x,t) - U'(\phi(x,t))$$



We will look for a static solution that goes from $\phi = -1$ at $x = -\infty$ to $\phi = +1$ at $x = +\infty$. [Dashen, Hasslacher, and Neveu]

Solving for the kink

We know how to solve $\phi''(x) = U'(\phi(x))$ in a different context: if x were t and $\phi(x)$ were $x(t)$, this would be ordinary Newtonian dynamics of a particle of unit mass in the potential $-U$.

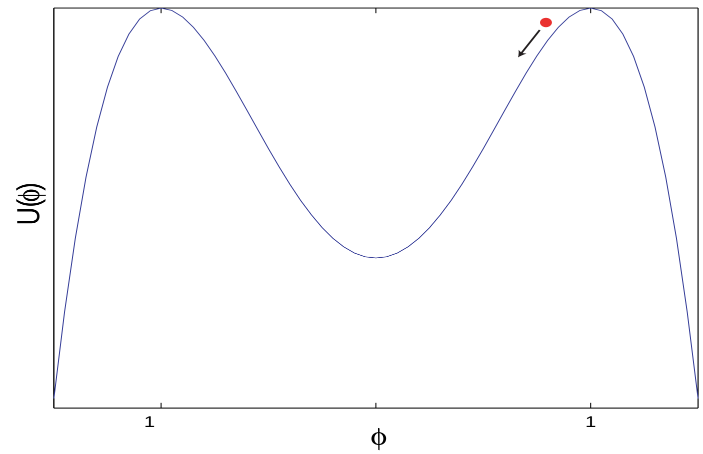
So the solution we are looking for "rolls" from one maximum of $-U$ to the other. Throughout this motion, its (conserved) "energy" is equal to zero:

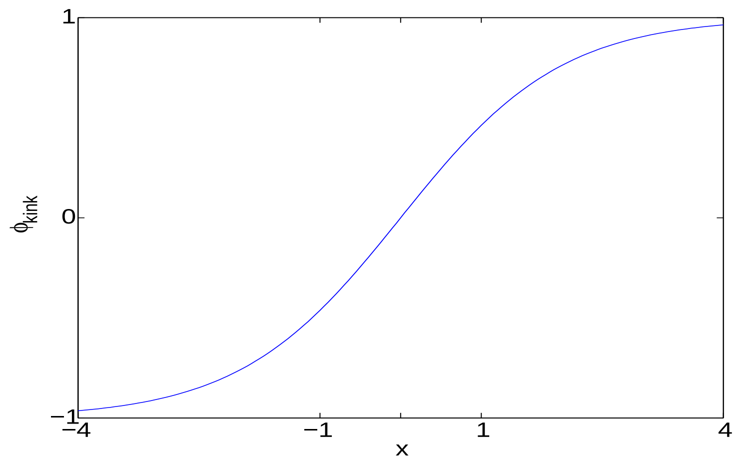
$$\frac{1}{2}\phi'(x)^2 - U(\phi) = 0.$$

So our kink (antikink) should have

$$\phi'_{\text{kink}}(x) = \pm\sqrt{2U(\phi_{\text{kink}})}$$

$$\Rightarrow \phi_{\text{kink}}(x) = \pm \tanh\frac{mx}{2}$$





Static localized solutions: Solitons

Many static solutions, like the kink, are the lowest energy configurations in a particular topological class, and thus are automatically stable against deformations. Other solitons are local minima of the energy without topological structure.

Other topological solitons include: the magnetic monopole in $SU(2)$ gauge theory with an adjoint Higgs ['t Hooft, Polyakov] and magnetic flux lines in superconductors [Abrikosov, Nielsen, Olesen].

Solitons typically carry such exotic charges and are of particular interest in the early universe (also string theory, condensed matter).

But they don't appear in every theory. For example, a scalar theory in more than one space dimension has no static solitons — they lower their energy by shrinking. [Derrick]

Time-dependent localized solutions: Oscillons/Breathers

More importantly, the Standard Model of particle physics has no known stable, localized, static classical solutions. (It does have instanton processes, and unstable Z-string [Vachaspati], and an unstable sphaleron [Manton and Klinkhamer].)

Solutions that are time-dependent but still localized evade Derrick's theorem and can exist in a wider variety of field theory models.

If solitons or oscillons form from a thermal background, they can provide a mechanism for sustained departures from equilibrium, which can be of particular interest in the early universe, especially baryogenesis, the formation of protons and neutrons we see today.

An integrable system

The "sine-Gordon" model is given by a slightly different potential, $U(\phi) = \frac{m^2}{2}(1 - \cos\phi)$, which similarly has static (anti)soliton solutions: $\phi(x) = 4\arctan e^{\pm mx}$.

This theory has an equivalent "dual" description in which the solitons are fundamental (fermionic) particles. [Coleman]

It is also integrable, with an infinite set of conserved charges. So we can solve its dynamics analytically. [Dashen, Hasslacher, and Neveu]

For example, collide a soliton and antisoliton. They pass right through each other with only a phase shift:

$$\phi(x,t) = 4\arctan\left(\frac{\sinh \gamma m u t}{u \cosh \gamma m x}\right)$$

where u is the incident speed and $\gamma = \frac{1}{\sqrt{1-u^2}}$.

An integrable system II

Letting $u = i/\epsilon$ we obtain an exact breather:

$$\phi(x,t) = 4\arctan\left(\frac{\epsilon\sin\gamma m t}{\cosh\gamma\epsilon m x}\right) \quad \text{where now} \quad \gamma = \frac{1}{\sqrt{1+\epsilon^2}}$$

- Temporal frequency is $\omega = \gamma m < m$.
- Spatial width is $\dfrac{1}{\kappa} = \dfrac{1}{m\gamma\epsilon}$.
- Amplitude is controlled by ϵ. For small ϵ, we can construct an approximate solution of this form for any potential, based on the leading nonlinear terms.
- At large distances, the field is small and a linear analysis holds: $\phi \approx 8\epsilon e^{-\kappa|x|}\sin\omega t$ with $\omega^2 = m^2 - \kappa^2$.

Q-balls: Stability via conserved charge

Q-balls are time-dependent solutions requiring only a single conserved charge, in a three-dimensional, nonintegrable scalar theory with no static solitons. The field in this model is complex. [Coleman]

We have the equation of motion

$$\ddot{\varphi}(\boldsymbol{x},t) = \nabla^2\varphi(\boldsymbol{x},t) - U'(\varphi(\boldsymbol{x},t))$$

where the potential is $U(\varphi) = \frac{1}{2}M^2|\varphi|^2 - A|\varphi|^3 + \lambda|\varphi|^4$.

There is a conserved charge

$$Q = \frac{1}{2i}\int d^3x \left(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*\right) .$$

We fix the charge Q by a Lagrange multiplier ω and obtain the Q-ball as a local minimum of the energy

$$\mathcal{E}_\omega[\varphi] = \int d^3x \left(\frac{1}{2}|\partial_t\varphi - i\omega\varphi|^2 + \frac{1}{2}|\nabla\varphi|^2 + U_\omega(|\varphi|)\right) + \omega Q$$

where $$U_\omega(|\varphi|) = \frac{1}{2}(M^2-\omega^2)|\varphi|^2 - A|\varphi|^3 + \lambda|\varphi|^4$$

Q-balls: Stability via conserved charge II

The Q-ball solution has simple time dependence: $\varphi(x,t) = e^{i\omega t}\phi(x)$.

We thus obtain the energy function

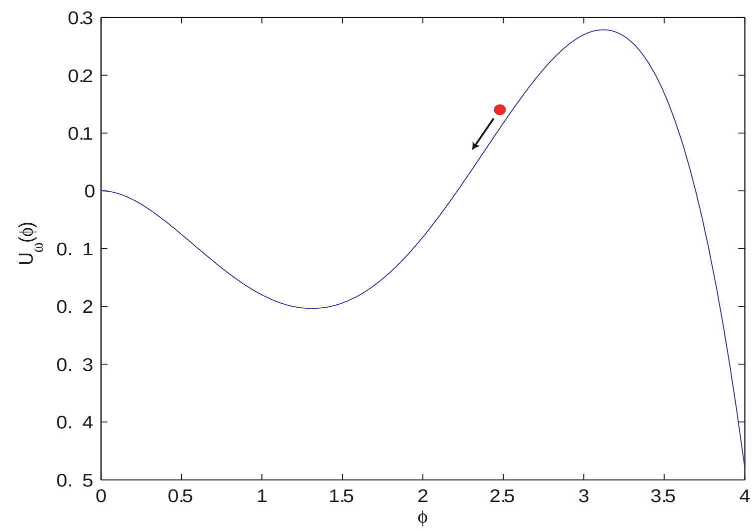
$$\mathcal{E}_\omega[\phi] = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}U_\omega(\phi)\right) + \omega Q$$

which is to be minimized over variations of ϕ and ω.

The equation for ϕ is

$$\frac{d^2}{dr^2}\phi(r) + \frac{2}{r}\frac{d}{dr}\phi(r) = U'_\omega(\phi)$$

which is again analogous to ordinary Newtonian mechanics, but now with "time"-dependent friction.



A simple overshoot/undershoot analysis shows that for a given ω, a solution exists with $\phi \to 0$ as $r \to \infty$. Then minimize this energy over ω to find the exact, periodic Q-ball solution.

Kink breathers: Forever = a very long time

Suppose we consider breathers, like we saw in the sine-Gordon model, but now for the ϕ^4 theory in $1+1$ dimensions. This model has static soliton solutions but does not have a useful conserved charge (we always have $\phi = 1$ at infinity), and is not integrable. So there are no simple expressions for exact breathers.

But for the right ranges of initial velocities, numerical simulations show breathers that live for an indefinitely long time.
[Campbell et. al.]

Breathers are stable to all orders in the multiple scale expansion.
[Dashen, Hasslacher, and Neveu]

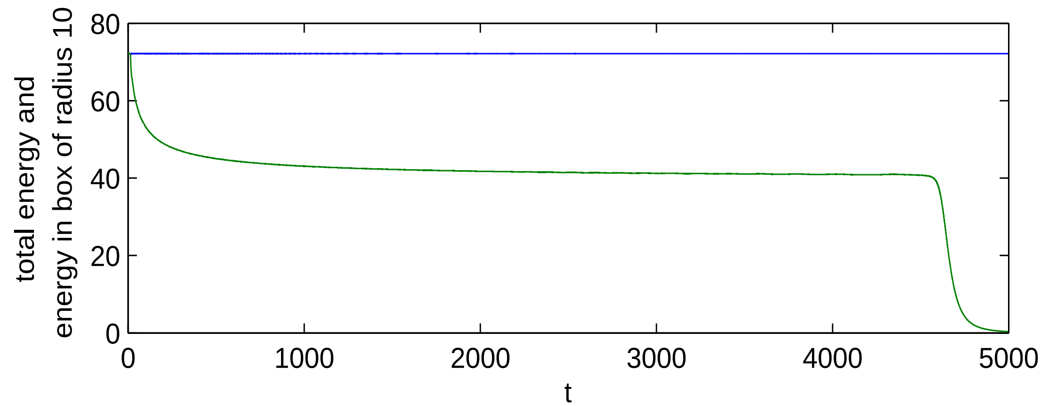
After much debate, the current consensus is that in the continuum, such configurations do decay, however, due to exponentially-suppressed non-perturbative effects. [Segur and Kruskal]

For physical applications this distinction is generally irrelevant.

ϕ^4 oscillons in three dimensions

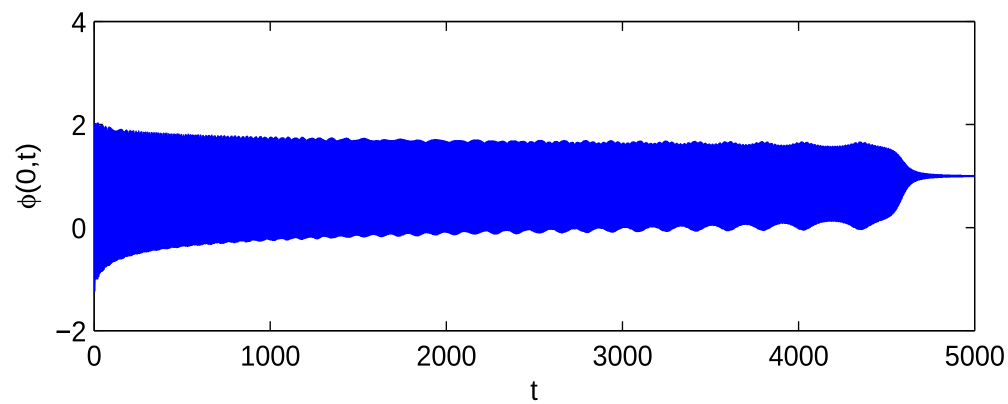
What about a real scalar ϕ^4 theory in three dimensions? It is not integrable, has no conserved charges, and no static solitons.

There are still (approximate) oscillons!



They live a long time, then suddenly decay.

[Gleiser]



Oscillon/breather heuristics

Q: Integrability, conserved charges, and the existence of static solitons all help us find oscillons, but none is necessary for them to exist. What is needed?

A: Nonlinearity and a frequency gap. The frequency of oscillation of the oscillon/breather is always below the frequency of the lowest linear mode, $\omega < m$. (For linear modes, $\omega = \sqrt{k^2 + m^2}$.)

The picture: nonlinearity allows oscillons/breathers to oscillate with a characteristic frequency that is too small to couple to the free dispersive waves in the system.

There are no outgoing modes available to dump their energy into. [Campbell et. al.]

Oscillon decay

Q: How does such a configuration decay?

A: By coupling to higher-frequency harmonics: 2ω, 3ω, etc.

If we cut off the high frequencies with a lattice such that $2\omega > \sqrt{m^2 + 4/(\Delta x)^2}$, then no such harmonics would exist, and the oscillon would be absolutely stable.

Even without this limitation, however, oscillons can live for an unnaturally long time.

Almost the Standard Model

Apply these ideas to the weak interactions in the Standard Model. We begin by ignoring fermions (matter), strong interactions, and electromagnetism. (Later we will restore electromagnetism.)

We have: ($\sigma^a =$ three generators of $SU(2)$ algebra)

- The weak interactions gauge field is a vector, $SU(2)$ adjoint:

$$A_\mu = (\, A_0 \quad A_1 \quad A_2 \quad A_3 \,) \quad \text{where} \quad A_\mu = A_\mu^a \frac{\sigma^a}{2}.$$

  Gives 3 real massive vector bosons ($W^\pm$ and Z^0, degenerate for us) with $m_W = \dfrac{gv}{2}$ (so linear waves have $\omega = \sqrt{k^2 + m_W^2}$).
  It's the analog of the electromagnetic field for weak interactions.

- The Higgs is a complex scalar, SU(2) fundamental: $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$.

  Gives 1 real massive scalar (Higgs boson) with $m_H = v\sqrt{2\lambda}$.
  They're currently searching for this at CERN.

Higgs and Gauge fields in the spherical ansatz

To make the problem tractable, we will write an ansatz for our field configurations that is as close to spherically symmetric as possible: It is invariant under simultaneous rotations of real space and isospin space. [Dashen, Hasslacher, and Neveu]

Write φ as a 2×2 matrix: $\Phi = \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix}$, so that $\Phi \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \varphi$.

The ansatz is:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{2g}\left(a_1(r,t)\hat{\boldsymbol{x}}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}}) + \frac{\alpha(r,t)}{r}(\boldsymbol{\sigma} - \hat{\boldsymbol{x}}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}})) + \frac{\gamma(r,t)}{r}(\hat{\boldsymbol{x}}\times\boldsymbol{\sigma}) \right) \\ A_0(\boldsymbol{x},t) &= \frac{1}{2g} a_0(r,t)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}} \qquad \Phi(\boldsymbol{x},t) = \frac{1}{g}\left(\mu(r,t) + i\nu(r,t)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}}\right) \end{aligned}$$

Ansatz is preserved under $U(1)$ gauge transformations:

$$A_\mu \to A_\mu - ig\left[\partial_\mu \Omega(r,t)\right](\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}}) \qquad \Phi \to \exp\left[i\Omega(r,t)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}}\right]\Phi$$

Effective 1-d theory

Form reduced fields in $1+1$ dimensions:

$$
\begin{aligned}
\phi(r,t) &= \mu(r,t) + i\nu(r,t) & \qquad D_\mu\phi = (\partial_\mu - \frac{i}{2}a_\mu)\phi \\
\chi(r,t) &= \alpha(r,t) + i(\gamma(r,t) - 1) & \qquad D_\mu\chi = (\partial_\mu - ia_\mu)\chi \\
a_\mu &= (\, a_0(r,t) \quad a_1(r,t)\,) & \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu
\end{aligned}
$$

where now $\mu, \nu = 0, 1$.

A $U(1)$ gauge theory! Mathematically similar to electromagnetism in one dimension (r), but with a rich set of interactions inherited from the full theory.

Gauge transformation:

$$
a_\mu \to a_\mu - i\partial_\mu\Omega(r,t) \qquad \phi \to e^{i\Omega(r,t)/2}\phi \qquad \chi \to e^{i\Omega(r,t)}\chi
$$

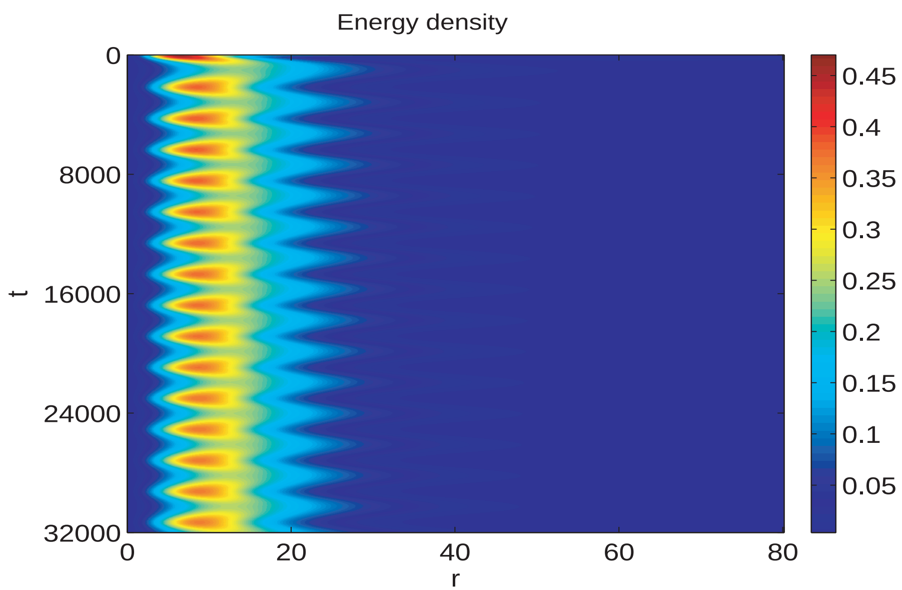
- ϕ has charge $1/2$ and mass m_H.
- χ has charge 1 and mass m_W.

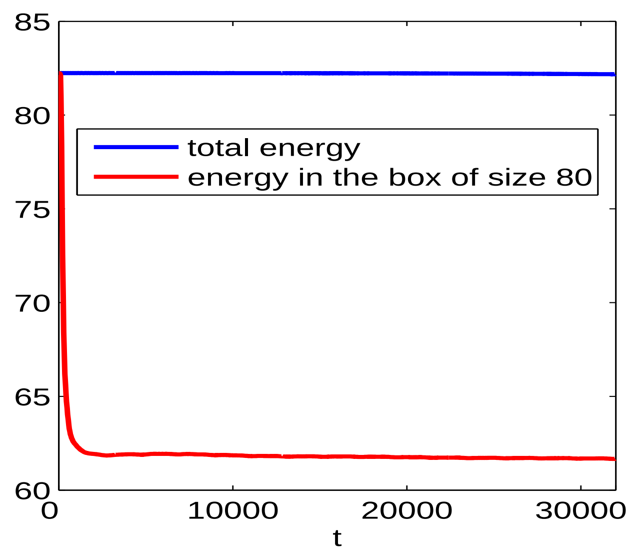
Spherical ansatz results

We do find oscillons in the spherical ansatz, but (so far) only if $m_H = 2m_W$.

In a further reduction of the spherical ansatz, this ratio can be explained using a small amplitude expansion.
[Stowell, Farhi, Graham, Guth, Rosales]

The oscillon is stable for as long as we can run numerically, with a "ringing" or "beat" pattern superimposed on the basic oscillations.





Electromagnetism returns

Now restore electromagnetism.

- Breaks isospin symmetry, so we won't stay in the spherical ansatz.
- Do a full 3-d simulation starting from spherical ansatz initial conditions, with no rotational symmetry assumptions. (We could also use an axially symmetric ansatz.)
- Z^0 is now split in mass from $W^{\pm}$.
- Massless photon a danger to oscillon stability.

Electroweak results

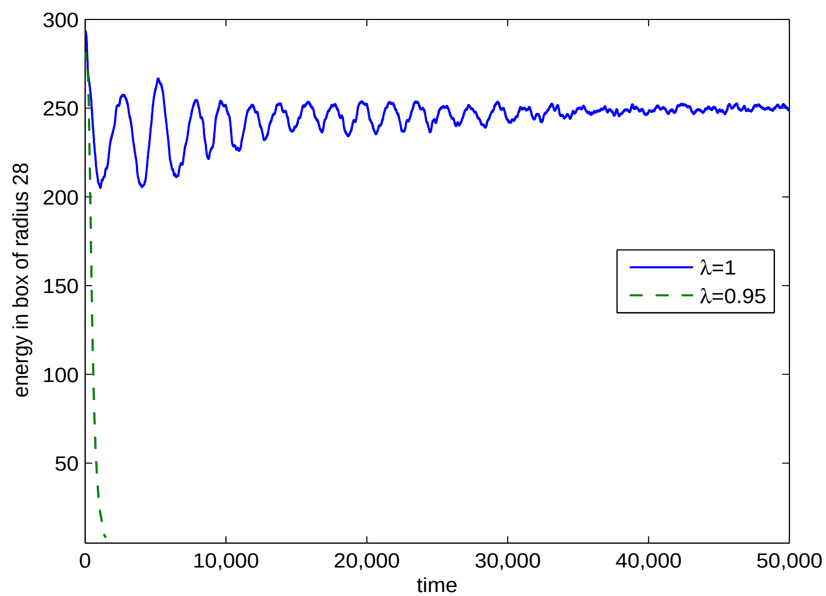
Spherical ansatz oscillons are modified but remain stable for $m_H = 2m_W$! (Not stable for $m_H = 2m_Z$.) [Graham]

Dangerous photon is disarmed because fields settle into an electrically neutral configuration.

"Beats" decay more rapidly with photon coupling included.

Observed solution has energy $E \approx 30$ TeV and size $r_0 \approx 0.05$ fm.

(1 mass unit $\approx$ 114 GeV.)



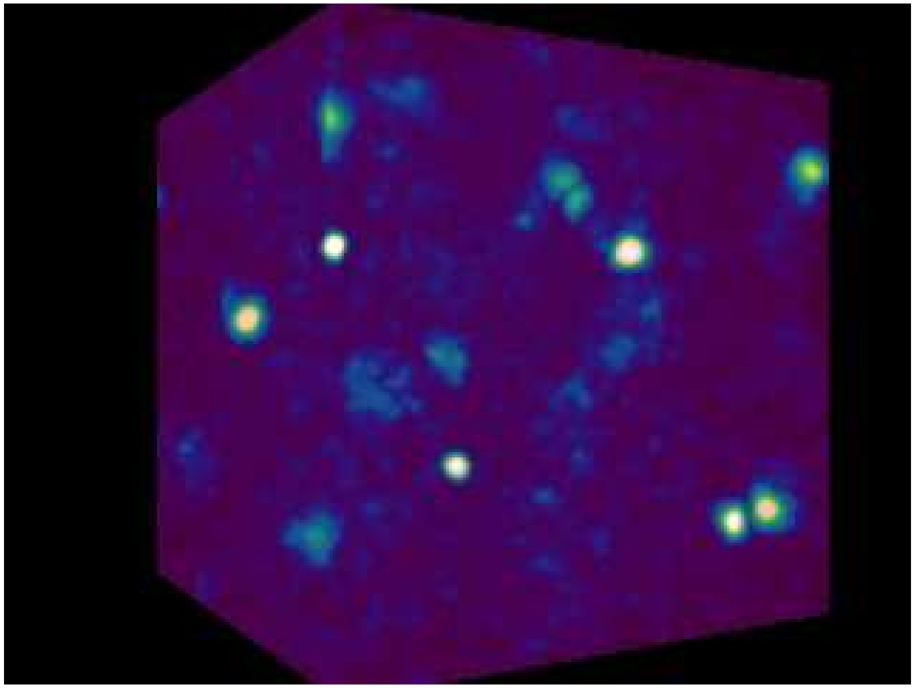
Conclusions

While solitons are easier to study, oscillons can appear in a wider range of theories.

Conserved charges, integrability and existence of static solitons are helpful for finding oscillons, but not necessary for oscillons to exist.

All oscillon solutions found numerically are attractors, or we never would have found them. Oscillons have been shown to form spontaneously from thermal initial conditions in an expanding universe.

[Farhi, Graham, Guth, Iqbal, Rosales, Stamatopoulos]



Even if oscillons are not perfectly stable, those that decay over "unnaturally" long time scales can be equally interesting.

Acknowledgements

Previous work done in collaboration with:	E. Farhi (MIT)
	V. Khemani (industry)
	R. Markov (Berkeley)
	R. R. Rosales (MIT)
Current collaborators on this project:	M. Gleiser (Dartmouth)
	N. Stamatopoulous (Dartmouth)
With support from:	National Science Foundation
	Research Corporation
	Vermont-EPSCoR
	Middlebury College