An Erdős-Stone Type Conjecture for Graphic Sequences

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Abstract

We consider a variation of the classical Turán-type extremal problem as introduced by Erdős, Jacobson and Lehel in [4]. Let π be an n-element graphic sequence and let H be a graph. We wish to determine the smallest even integer m such that any n-term graphic sequence π having degree sum at least m has *some* realization containing H as a subgraph. Denote this value m by $\sigma(H, n)$. For an arbitrarily chosen H, we construct a graphic sequence $\pi(H, n)$ whose degree sum plus two is at least $\sigma(H, n)$. Furthermore, we conjecture that equality holds in general, as this is the case for all choices of H where $\sigma(H, n)$ is currently known.

Keywords: Degree sequence, Potentially graphic sequence

1 Introduction

We consider only simple undirected graphs. We will denote the complete graph on t vertices by K_t, and the join of two graphs, G and H, by $G + H$. Further, we will write $H \subset G$ if H is a subgraph of G and $H < G$ if H is an induced subgraph of G.

A sequence of nonnegative non-increasing integers $\pi = (d_1, d_2, \ldots, d_n)$ is said to be *graphic* if there exists a graph G of order n having degree sequence π. The graph G is said to be a *realization* of π. Writing $\pi = (d_1^{\mu_1}, \ldots, d_t^{\mu_t})$ will indicate that d_i repeats μ_i times.

For a given graph H, a sequence π is said to be *forcibly H-graphic* if every realization of π contains H as a subgraph. Thus Turán's classical question [22] may be re-phrased as follows: determine the least even integer M so that every n-term graphic sequence with the sum of the terms exceeding M is forcibly K_t-graphic. The same question for an arbitrary graph H was considered and solved by Erdős-Stone [6] (see also Erdős and Simonovits [5]). Sometimes referred to as the fundamental theorem of extremal graph theory, their result states that this value is determined by the chromatic number of H and n.

For a given graph H, a sequence π is said to be *potentially H-graphic* if there exists *some* realization of π which contains H as a subgraph. In the early 1990's, Erdős, Jacobson and Lehel [4] posed the following problem: determine the least even integer m so that every n-term graphic sequence with the sum of the terms exceeding m is potentially K_t-graphic. They proposed that this value, denoted $\sigma(K_t, n)$, is $(t-2)(2n-t+1)+2$ as they considered the degree sequence $((n-1)^{t-2}, (t-2)^{n-t+2})$ as the extremal sequence. (The unique realization of this sequence is $K_{t-2} + \overline{K}_{n-t+2}$.) This value was shown to be correct for the cases $t = 3, 4$ and 5 (see respectively [4], [11] and [16], and [17]), and Li, Luo and Song [18] proved the conjecture true via linear algebraic techniques for $t \geq 6$ and $n \geq \binom{t}{2} + 3$. A purely graph-theoretic proof was given in [8] and also as a corollary to the main result in [3].

The aim of this note is, for an arbitrarily chosen graph H, to give a lower bound for $\sigma(H, n)$, that is to determine the least positive even integer so that every n-term graphic sequence exceeding this $\sigma(H, n)$ is potentially H-graphic.

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2 Main Result

We assume that H has no isolated vertices and that n is sufficiently large with respect to the number of vertices of H, $|V(H)|$. Let $\alpha := \alpha(H)$ denote the independence number of H. We define the quantities

$$u := u(H) = |V(H)| - \alpha(H) - 1,$$

and

$$d := d(H) = min\{\Delta(F) : F < H, |V(F)| = \alpha(H) + 1\}.$$

Let $v_i(H)$ denote the number of vertices of degree i in H. For all $i, d \leq i \leq \alpha - 1$ we define the quantity m_i to be the minimum number of vertices of degree i over all induced subgraphs F of H with $|V(F)| = \alpha + 1$. The quantities n_i, $d \leq i \leq \alpha - 1$ are defined recursively such that $n_d = m_d - 1$ and $n_i = \min\{m_i - 1, n_{i-1}\}$. Finally, we define $\delta_{\alpha-1} = n_{\alpha-1}$ and for $d \leq i \leq \alpha - 2$ we define $\delta_i = n_i - n_{i+1}$.

We now consider the following sequence:

$$\pi(H, n) = ((n-1)^u, (u+\alpha-1)^{\delta_{\alpha-1}}, (u+\alpha-2)^{\delta_{\alpha-2}}, \ldots \\ \ldots (u+d)^{\delta_d}, (u+d-1)^{n-u-\Sigma\delta_i}).$$

If this sequence is not graphic, then we will reduce the smallest term which is strictly greater than $u(H)$ in the sequence by one and redefine $\pi(H, n)$ to be this graphic sequence instead. Let $\sigma(\pi(H, n))$ denote the sum of the terms of $\pi(H, n)$.

Theorem 2.1 *([10]) Given a graph H and n sufficiently large then,*

$$\sigma(H, n) \geq \max\{\sigma(\pi(H^*, n)) + 2 \mid H^* \subseteq H\}. \tag{1}$$

It has been shown that equality holds in Theorem 2.1 for the following graphs: complete graphs [4], [11], [16], [17], [18], complete bipartite graphs [2],[11], [19], complete balanced multipartite graphs [8], [1], matchings [11], cycles [13], the friendship graph $(K_1 + kK_2)$ [9],[12], split graphs [3], a disjoint union of cliques [7] and a complete graph with an edge deleted [14], [15]. At this time we know of no subgraph for which equality does not hold and thus conjecture that the inequality sign in Equation 1 may be replaced by an equality sign.

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